A modal account of preference in a fuzzy setting

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Abstract. In this paper we first consider the problem of extending fuzzy (weak and strict) preference relations on a set, represented by fuzzy preorders, to a fuzzy preferences on subsets, and we characterise different possibilities. Based on their properties, we then semantically define and axiomatize several two-tiered graded modal logics to reason about the corresponding different notions of fuzzy preferences.

Keywords: preference structures, fuzzy preorder, strict fuzzy order, preference two-tiered modal logic

1 Introduction

Reasoning about preferences is a hot topic in Artificial Intelligence since many years, see for instance [17, 5, 18]. Two main approaches for representing and handling preferences have been developed: the relational and the logic-based approaches.

In classical preference relations, every preorder R (and more in general every reflexive relation) can be regarded as a preference relation by assuming that $(a, b) \in R$ means that *a is preferred or indifferent to b*. From R we can define three disjoint relations:

- the *strict preference* $P = R \cap R^d$,
- the *indifference relation* $I = R \cap R^t$, and
- the *incomparability relation* $J = R^c \cap R^d$.

where $R^d = \{(a, b) \in R : (b, a) \notin R\}$, $R^c = \{(a, b) \in R : (b, a) \in R\}$ and $R^t = \{(a, b) : (b, a) \in R\}$. It is clear that P is a strict order (irreflexive, antisymmetric and transitive), I is an equivalence relation (reflexive, symmetric and transitive) and J is irreflexive and symmetric. The triple (P, I, J) is called a *preference structure*, where the initial weak preference relation can be recovered as $R = P \cup I$.

In the fuzzy setting, preference relations can be attached degrees (usually belonging to the unit interval $[0, 1]$) of fulfilment or strength, so they become *fuzzy relations*. A weak fuzzy preference relation on a set X will be now a fuzzy preorder $R : X \times X \to [0, 1]$, where $R(a, b)$ is interpreted as the degree in which b is at least as preferred as a. Given a t-norm $\odot$, a fuzzy $\odot$-preorder satisfies reflexivity ($R(a, a) = 1$ for each $a \in X$) and $\odot$-transitivity

($R(a,b) \odot R(b,c) \leqslant R(a,c)$ for each $a, b, c \in X$). The most influential reference is the book by Fodor and Roubens [6], that was followed by many other works like, for example [7–11]. The problem in this setting is how to define the corresponding strict preference, indifference and incomparability relations from the initial fuzzy preorder. Many questions arise since it is possible to generalise the classical case in many different ways. In particular, several works have paid attention to how suitably interrelate a weak preference (a fuzzy preorder) with its associated indifference relation (a indistinguishability relation) and strict preference (a strict fuzzy order). In this sense, relevant publications are, among others, Bodenhofer's papers [2–4]. There, the author studies $\odot$-E fuzzy preorders related to a t-norm $\odot$ and an indistinguishability, or *fuzzy equivalence*, relation E (reflexive, symmetric and $\odot$-transitive), as well as their strict associated fuzzy orders in a general context, which is also applies to the context of preference modelling. Indeed, given a t-norm $\odot$ and an indistinguishability relation E, a $\odot$-E fuzzy preorder is defined as a fuzzy relation $R : X \times X \to [0,1]$ satisfying: E*-reflexivity*: $R(x,y) \geqslant E(x,y)$, $\odot$-E*-antisymmetry*: $R(x,y) \odot R(y,x) \leqslant E(x,y)$, $\odot$*-transitivity*: $R(x,y) \odot R(y,z)) \leqslant R(x,z)$. Bodenhofer also studies how to extend such a $\odot$-E fuzzy preorder to the set $\mathcal{F}(\mathcal{X})$ of fuzzy subsets of a universe X, as well as the associated indistinguishability relation and the strict fuzzy order, and discusses different possible definitions. In such a setting, he considers both the cases of crisp and fuzzy preorders, but he does not consider the particular case we will study in this paper, namely the interaction of a fuzzy preferences over crisp subsets of X.

The basic assumption in logical approaches is that preferences have structural properties that can be suitably described in a fomalized language.This is the main goal of the so-called *preference logics*, see e.g. [17]. The first logical systems to reason about preferences go back to S. Halldén [20] and to von Wright [22, 23, 16]. More recently van Benthem et al. in [1] have presented a modal logic-based formalization of preferences. In that paper the authors first define a basic modal logic with two unary modal operators $\Diamond^{\leqslant}$ and $\Diamond^{<}$, together with the universal and existential modalities, A and E respectively, and axiomatize it. Using these primitive modalities, they consider several (definable) binary modalities to capture different notions of preference relations on classical propositions, and show completeness with respect to the intended preference semantics. Finally they discuss their systems in relation with von Wright axioms for *ceteris paribus* preferences [22]. On the other hand, with the motivation of formalising a comparative notion of likelihood, Halpern studies in [15] different ways to extend preorders on a set X to preorders on subsets of X and their associated strict orders. He studies their properties and relations among them, and he also provides an axiomatic system for a *logic of relative likelihood*, that is proved to be complete with respect to what he calls *preferential structures*, i.e. Kripke models with preorders as accessibility relations.

In this paper we begin by studying in Section 2 different forms to define fuzzy relations on the set $\mathcal{P}(\mathrm{W})$ of subsets of W, from a fuzzy preorder on W, in a similar way to the one followed in [1, 15] for classical preorders, and in [2, 3] for fuzzy preorders. In Section 3 we characterize them and discuss which are the most appropriate from the point of view of preference modelling, while in Section 4 we deal with the problem of defining a fuzzy strict order in a set associated to a given fuzzy preorder, and how to lift them to susbsets. Finally, in Section 5, and based on the previous results, we semantically define and axiomatize several two-tiered graded modal logics to reason about different notions of preferences.

This paper is a proper extended version of the conference paper [13].

2 Extending a fuzzy preorder on a set W to a fuzzy relation on subsets of W

2.1 Precedents in the classical case

In the classical logic setting, van Benthem et al. define in [1] *preference models* as triples $\mathcal{M} = (\mathrm{W}, \preceq, \mathcal{V})$ where W is a set of states or worlds, $\preceq$ is a preorder (reflexive and transitive) relation on W, and $\mathcal{V}$ is a standard propositional evaluation, that is, a mapping assigning to every propositional variable p a subset $\mathcal{V}(p) \subseteq \mathrm{W}$ of states where p is true. $\mathcal{V}$ can be extended to any propositional formula φ by using the classical Boolean definitions. For simplicity, we will also denote $\mathcal{V}(\varphi)$ by $[\varphi] = \{w \in W : w(\varphi) = 1\}$.

Then they consider the following four binary preference operators on propositions.

Definition 1 (cf. [1]). *Given a preference model $\mathcal{M} = (\mathrm{W}, \preceq, \mathcal{V})$, one can define the following four binary preference operators on classical propositions:*

- $\mathcal{M} \models \varphi \leqslant_{\exists\exists} \psi$ *iff there exist* $u \in [\varphi], v \in [\psi]$ *such that* $u \leqslant v$.
- $\mathcal{M} \models \varphi \leqslant_{\exists\forall} \psi$ *iff there exists* $u \in [\varphi]$*, such that for all* $v \in [\psi]$*,* $u \leqslant v$.
- $\mathcal{M} \models \varphi \leqslant_{\forall\exists} \psi$ *iff for all* $u \in [\varphi]$*, there exists* $v \in [\psi]$ *such that* $u \leqslant v$.
- $\mathcal{M} \models \varphi \leqslant_{\forall\forall} \psi$ *iff for all* $u \in [\varphi]$ *and* $v \in [\psi]$*, then* $u \leqslant v$.

Notice that these definitions of the truth conditions for the four preference operators can be interpreted as defining corresponding preference relations on $\mathcal{P}(\mathrm{W})$, the power set of W (which contains the sets $[\varphi]$) arising from a preorder on the set of worlds W. One can furthermore define two more preference operators on propositions:

- $\mathcal{M} \models \varphi \leqslant_{\exists\forall 2} \psi$ iff there exists $v \in [\psi]$, such that for all $u \in [\varphi]$, $u \leqslant v$
- $\mathcal{M} \models \varphi \leqslant_{\forall\exists 2} \psi$ iff for all $v \in [\psi]$, there exists $u \in [\varphi]$ such that $u \leqslant v$.

Therefore, from a given preorder on W we can consider six relations on subsets of W. The basic set-inclusions between these relations are given in the following proposition.

Proposition 1. *The following inclusions hold:*

$$\leqslant_{\forall\forall} \subseteq \leqslant_{\forall\exists} \subseteq \leqslant_{\exists\exists}, \quad \leqslant_{\forall\forall} \subseteq \leqslant_{\exists\forall} \subseteq \leqslant_{\exists\exists}, \quad \leqslant_{\forall\forall} \subseteq \leqslant_{\forall\exists 2} \subseteq \leqslant_{\exists\exists}, \quad \leqslant_{\forall\forall} \subseteq \leqslant_{\exists\forall 2} \subseteq \leqslant_{\exists\exists}$$

Moreover, the four intermediate relations are not comparable, except for the following inclusions:

$$\leqslant_{\exists\forall 2} \subseteq \leqslant_{\forall\exists}, \quad \leqslant_{\exists\forall} \subseteq \leqslant_{\forall\exists 2} .$$

Proof. All the inclusion relations are easy to check. Moreover, the inclusions given in Proposition 1 are the only ones that are valid among the four intermediate relations, as the following examples show: Take $W = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $A = \{u_1, u_2, u_3\}, B = \{u_4, u_5, u_6\}$. Then,

- If the preorder is defined by reflexivity plus $u_1 \leqslant u_4, u_2 \leqslant u_5$ and $u_3 \leqslant u_5$, then $A \leqslant_{\forall\exists} B$ is the unique intermediate relation that is satisfied.
- If the preorder is defined by reflexivity plus $u_1 \leqslant u_4, u_2 \leqslant u_5$ and $u_2 \leqslant u_6$, then $A \leqslant_{\forall\exists 2} B$ is the unique intermediate relation that is satisfied.
- If the preorder is defined by reflexivity plus $u_2 \leqslant u_4, u_2 \leqslant u_5$ and $u_2 \leqslant u_6$, then $A \leqslant_{\exists\forall} B$ and $A \leqslant_{\forall\exists 2} B$ are the unique intermediate relations that are satisfied.
- If the preorder is defined by reflexivity plus $u_1 \leqslant u_4, u_2 \leqslant u_4$ and $u_3 \leqslant u_4$, then $A \leqslant_{\exists\forall 2} B$ and $A \leqslant_{\forall\exists} B$ are the unique intermediate relations that are satisfied.

2.2 The fuzzy preorder case

Now we study the case when $\leqslant$ is a fuzzy $\odot$-preorder on W, i.e., $\leqslant: \mathrm{W} \times \mathrm{W} \longrightarrow [0,1]$ satisfying reflexivity ($[u \leqslant u] = 1$ for all $u \in \mathrm{W}$) and $\odot$-transitivity with respect to a given t-norm $\odot$ (for all $u, v, w \in \mathrm{W}$, $([u \leqslant v] \odot [v \leqslant w]) \leqslant [u \leqslant w]$), where $[u \leqslant v]$ denotes the value in $[0,1]$ of the fuzzy relation $\leqslant$ applied to the ordered pair of elements $u, v \in \mathrm{W}$. We will assume that W is a *finite* set, and we will denote by δ_u the singleton $\{u\}$.

Generalising the classical case, we can define the following fuzzy relations on $\mathcal{P}(\mathrm{W})$ from a fuzzy preorder on W.

Definition 2. *Given a fuzzy preorder $\leqslant$ on* W*, we can define the following six fuzzy relations on $\mathcal{P}(\mathrm{W})$. For any $A, B \in \mathcal{P}(\mathrm{W})$ we let:*

- $[A \leqslant_{\exists\exists} B] = \sup_{u \in A} \sup_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\exists\forall} B] = \sup_{u \in A} \inf_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\exists} B] = \inf_{u \in A} \sup_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\forall} B] = \inf_{u \in A} \inf_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\exists 2} B] = \inf_{v \in B} \sup_{u \in A} [u \leqslant v]$
- $[A \leqslant_{\exists\forall 2} B] = \sup_{v \in B} \inf_{u \in A} [u \leqslant v]$.

where the value of $A \leqslant_{\circ} B$ is denoted by $[A \leqslant_{\circ} B]$ with $\leqslant_{\circ}$ being anyone of the six relations.

It is clear that, since the preorder $\leqslant$ is valued on $[0,1]$, these relations are also $[0,1]$-valued. For each $a \in (0,1]$, we will write $A \leqslant^{a}_{\exists\exists} B$ when $[A \leqslant_{\exists\exists} B] \geqslant a$ and analogously for the other relations.

Proposition 2. *For any sets $A, B \in \mathcal{P}(\mathrm{W})$, we have:*

- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\forall\exists} B] \leqslant [A \leqslant_{\exists\exists} B]$,
- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\forall\exists 2} B] \leqslant [A \leqslant_{\exists\exists} B]$,

– $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\exists\forall} B] \leqslant [A \leqslant_{\exists\exists} B]$, *and*
– $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\exists\forall 2} B] \leqslant [A \leqslant_{\exists\exists} B]$.

Moreover the four intermediate relations are not comparable, except for the same two cases (now inequalities) of Prop. 1.

Proof. Analogous to the proof of Prop. 1.

Out of the above six possibilities, we will mainly focus on two of them, $\leqslant_{\forall\exists}$ and $\leqslant_{\forall\exists 2}$, in the rest of the paper. These are well-behaved extensions of an initial fuzzy $\odot$-preorder to model a weak preference relation on subsets, since in particular they keep being $\odot$-preorders. Moreover, combining them, we can capture a very natural (preference) ordering related to orderings of intervals. Indeed, suppose $(W, \leqslant)$ is a totally (classical) pre-ordered set, and we want to extend $\leqslant$ to an ordering on the set $Int(\mathrm{W})$ of intervals of W. The two most usual ways to do this are the following:

(i) $[a, b] \leqslant_1 [c, d]$ if $a \leqslant c$ and $b \leqslant d$,
(ii) $[a, b] \leqslant_2 [c, d]$ if $b \leqslant c$.

The relation $\leqslant_1$ is considered for example in [2], and it turns out to be definable as the intersection of the $\leqslant_{\forall\exists}$ and $\leqslant_{\forall\exists 2}$ relations on $Int(\mathrm{W})$, that is, $\leqslant_1 = \leqslant_{\forall\exists} \cap \leqslant_{\forall\exists 2}$, while the second, $\leqslant_2$, coincides with the (crisp) relation $\leqslant_{\forall\forall}$ on $Int(A)$. Actually, $\leqslant_{\forall\forall}$ is not a preorder because it is only reflexive for singletons, but it is enough for our purposes. In next sections, we will study in the fuzzy case these three basic relations ($\leqslant_{\forall\exists}$, $\leqslant_{\forall\exists 2}$, $\leqslant_{\forall,\forall}$) on $\mathcal{P}(\mathrm{W})$ arising from a fuzzy preorder $\leqslant$ on W.

3 Characterizing the relations $\leqslant_{\forall\exists}$, $\leqslant_{\forall\exists 2}$ and $\leqslant_{\forall\forall}$

The following propositions describe the main properties satisfied by each one of these relations. In what follows, we assume a given a fuzzy $\odot$-preorder $\leqslant$ on W and the fuzzy relations $\leqslant_{\forall\exists}, \leqslant_{\forall\exists 2}$ and $\leqslant_{\forall\forall}$ which are defined as in Definition 2.

Proposition 3. *The relation $\leqslant_{\forall\exists}$ satisfies the following properties, for all $A, B, C \in \mathcal{P}(\mathrm{W})$:*

1. *Inclusion:* $[A \leqslant_{\forall\exists} B] = 1$, *if* $A \subseteq B$
2. $\odot$*-Transitivity:* $[A \leqslant_{\forall\exists} B] \odot [B \leqslant_{\forall\exists} C] \leqslant [A \leqslant_{\forall\exists} C]$
3. *Left-OR:* $[(A \cup B) \leqslant_{\forall\exists} C] = \min([A \leqslant_{\forall\exists} C], [B \leqslant_{\forall\exists} C])$
4. *Restricted Right-OR:* $[A \leqslant_{\forall\exists} (B \cup C)] \geqslant \max([A \leqslant_{\forall\exists} B], [A \leqslant_{\forall\exists} C])$. *The inequality becomes an equality if A is a singleton.*

Proposition 4. *The relation $\leqslant_{\forall\exists 2}$ satisfies the following properties, for all $A, B, C \in \mathcal{P}(\mathrm{W})$:*

1. *Inclusion:* $[A \leqslant_{\forall\exists 2} B] = 1$, *if* $B \subseteq A$
2. $\odot$*-Transitivity:* $[A \leqslant_{\forall\exists 2} B] \odot [B \leqslant_{\forall\exists 2} C] \leqslant [A \leqslant_{\forall\exists 2} C]$

3. *Restricted Left-OR:* $[(A \cup B) \leqslant_{\forall\exists 2} C] \geqslant \max([A \leqslant_{\forall\exists 2} C], [B \leqslant_{\forall\exists 2} C])$. *The inequality becomes an equality if C is a singleton.*
4. *Right-OR:* $[A \leqslant_{\forall\exists 2} (B \cup C)] = \min([A \leqslant_{\forall\exists 2} B], [A \leqslant_{\forall\exists 2} C])$.

Proposition 5. *The relation $\leqslant_{\forall\forall}$ satisfies the following properties, for all $A, B, C \in \mathcal{P}(\mathrm{W})$:*

1. *Restricted reflexivity:* $[A \leqslant_{\forall\forall} A] = 1$ *iff A is a singleton*
2. *$\odot$-Transitivity:* $[A \leqslant_{\forall\forall} B] \odot [B \leqslant_{\forall\forall} C] \leqslant [A \leqslant_{\forall\forall} C]$
3. *Left-OR:* $[(A \cup B) \leqslant_{\forall\forall} C] = \min([A \leqslant_{\forall\forall} C], [B \leqslant_{\forall\forall} C])$
4. *Right-OR:* $[A \leqslant_{\forall\forall} (B \cup C)] = \min([A \leqslant_{\forall\forall} B], [A \leqslant_{\forall\forall} C])$
5. *Inclusions:* $[A \leqslant_{\forall\forall} B] \leqslant [A' \leqslant_{\forall\forall} B']$, *if $A' \subseteq A, B' \subseteq B$.*

The proofs of the these propositions are easy and we omit them. Observe that, as already mentioned above, $\leqslant_{\forall\forall}$ is not reflexive.

Actually, the properties given above fully characterize the different relations on $\mathcal{P}(\mathrm{W})$ as showed in the next theorem.

Theorem 1. *The following characterizations hold:*

(i) *Let $\preceq_{AE}$ be a relation between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 1, 2, 3 and 4 of Prop. 3. Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\preceq_{AE}$ coincides with $\leqslant_{\forall\exists}$ as defined in Def. 2.*

(ii) *Let $\preceq_{AE2}$ be a relation between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 1, 2, 3 and 4 of Prop. 4. Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\preceq_{AE2}$ coincides with $\leqslant_{\exists\forall 2}$ as defined in Def. 2.*

(iii) *Let $\preceq_{AA}$ be a relation between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 1, 2, 3, 4 and 5 of Prop. 5. Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\preceq_{AA}$ coincides with $\leqslant_{\forall\forall}$ as defined in Def. 2.*

Proof. We show the case of $\leqslant_{AE}$, the rest of cases are proved in a similar way. Define a relation on W by $[u \leqslant v] = [\delta_u \preceq_{AE} \delta_v]$. Clearly $\leqslant$ is a fuzzy preorder on W. Now take into account that, for all $A \in \mathcal{P}(\mathrm{W})$, $A = \bigcup\{\delta_u : u \in A\}$ and, applying Properties 3 and 4, it is obvious that for all $A, B \in \mathrm{W}$, then $[A \preceq_{AE} B] = \inf_{u \in A} \sup_{v \in B} [\delta_u \preceq \delta_v]$. Thus (i) is proved. □

4 Characterizing strict fuzzy orders associated to fuzzy preorders

It is well known that any crisp preorder $\leqslant$ on an universe W induces an equivalence (or indifference) relation $\equiv$ and an strict order $<$, defined as follows:

- $x \equiv y$ iff $x \leqslant y$ and $y \leqslant x$,
- $x < y$ iff $x \leqslant y$ and $x \neq y$ or, alternatively iff $x \leqslant y$ and $y \nleqslant x$.

Observe that these relations satisfy that $x \leqslant y$ iff either $x \equiv y$ or $x < y$. We will use this condition to define an strict fuzzy order associated to a fuzzy preorder.

In the fuzzy setting (see for example [2, 14]), from a fuzzy $\odot$-preorder $\leqslant$: $W \times W \to [0, 1]$ we can define :

- the maximal indistinguishability relation $v \equiv w$ contained in the fuzzy preorder, defined by $[x \equiv y] = [x \leqslant y] \wedge [y \leqslant x]$;
- the minimal strict fuzzy $\odot$-order $<$ that satisfies the following equation

$$[x \leqslant y] = [x < y] \oplus [x \equiv y] \tag{1}$$

where $\oplus$ is a T-conorm (for example the maximum or the bounded sum).

So defined, the relation $\equiv$ is reflexive, symmetric and $\odot$-transitive, and thus it is a $\odot$-indistinguishability relation (the generalization of the crisp notion of equivalence relation). On the other hand, the minimal solution for b of the equation $a \leqslant b \oplus c$ in $[0,1]$, is the so-called dual resituated implication, or implication associated to the T-conorm $\oplus$, which is defined as,

$$a \to^{\oplus} c = \inf\{b \mid b \oplus a \geqslant c\}.$$

Therefore, we take as the strict fuzzy order relation $<$ associated to $\leqslant$ for the T-conorm $\oplus$ the fuzzy relation defined as

$$[x < y] = [x \leqslant y] \to^{\oplus} [x \equiv y] = [x \leqslant y] \to^{\oplus} [y \leqslant x].$$

An easy computation shows that the strict fuzzy order relation for $\oplus = \max$ is defined as

$$[x < y] = \begin{cases} [x \leqslant y], & \text{if } [x \leqslant y] > [y \leqslant x], \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

And for $\oplus$ being the bounded sum (i.e. the Łukasiewicz T-conorm) is[3]

$$[x < y] = \begin{cases} [x \leqslant y] - [y \leqslant x], & \text{if } [x \leqslant y] > [y \leqslant x], \\ 0, & \text{otherwise.} \end{cases}$$

The strict relation associated to $\leqslant$ is a irreflexive ($[x < x] = 0$) and antisymmetric ($\min([x < y], [y < x]) = 0$) but, as far as we know, it is not known whether it is $\odot$-transitive in general. Nevertheless we have the following result.

Proposition 6. *Let $\leqslant$ be a* min*-preorder on a universe* W *and let $<$ be the associated strict relation w.r.t. $\oplus = \max$. Then $<$ is* min*-transitive.*

Proof. The proof is by contradiction. Suppose the strict relation is not min-transitive. Then there must exist elements $u, v, w \in W$ such that $[u < v], [v, w] > 0$ and $[u < w] = 0$ which is equivalent that $[u \leqslant v] = a > b = [v \leqslant u]$, $[v \leqslant w] = c > d = [w \leqslant v]$ and $[u \leqslant w] = [w \leqslant u] = f$. Thus we have five values a, b, c, d, f and we know that

$$a > b \text{ and } c > d. \tag{*}$$

We can now reason by cases:

[3] This is the strict order companion defined and studied in [7].

(1) Suppose $a \geqslant c$ and $b \geqslant d$. Combining this assumption with (*) we have that $a \geqslant c > d$. By transitivity, $f \geqslant \min(a, c) = c$ and $f \geqslant \min(d, b) = d$ by hypothesis. Moreover $\min([w \leqslant u], [u \leqslant v]) = \min(f, a) \leqslant d = [w \leqslant v]$. This implies that $a \leqslant d$, in contradiction with the fact that $d < a$.
(2) Suppose $a \geqslant c$ and $b < d$. Combining this assumption with (*) we have that $d < c \leqslant a$. By transitivity, $f \geqslant \min(a, c) = c$ and $f \geqslant \min(d, b) = b$ by hypothesis. Moreover $\min([w \leqslant u], [u \leqslant v]) = \min(f, a) \leqslant d = [w \leqslant v]$. This implies that $f \leqslant d$, and by hypothesis $f \leqslant d < c$, in contradiction with $f \geqslant c$ previously proved.
(3) Suppose $a \leqslant c$ and $b \geqslant d$. Combining this assumption with (*) we have that $b < a \leqslant c$. By transitivity, $f \geqslant \min(a, c) = a$ and $f \geqslant \min(d, b) = d$ by hypothesis. Moreover $\min([v \leqslant w], [w \leqslant u]) = \min(c, f) \leqslant b = [v \leqslant u]$. This imply that $f \leqslant b$ and by hypothesis $f \leqslant b < a$, in contradiction with $f \geqslant a$ previously proved.
(4) Suppose $a \leqslant c$ and $b \leqslant d$. Combining this assumption with (*) we have that $b < a \leqslant c$. By transitivity, $f \geqslant \min(a, c) = a$ and $f \geqslant \min(d, b) = b$ by hypothesis. Moreover $\min([v \leqslant w], [w \leqslant u]) = \min(c, f) \leqslant b = [v \leqslant u]$. This implies that $f \leqslant b$, and by hypothesis $f \leqslant b < a$, in contradiction with $f \geqslant a$ previously proved. □

From now on, we consider the strict fuzzy order $<$ associated to $\leqslant$ the one defined by taking $\oplus = \max$ according to (2).

Now we can come back to the topic of how to define a strict fuzzy order relation on sets of $\mathcal{P}(\mathrm{W})$ corresponding to a fuzzy preorder in W. Halpern notices in [15] that there are two different ways to define (in the crisp case) a strict relation on $\mathcal{P}(\mathrm{W})$ from a preorder on W. The same idea applied to the fuzzy case gives rise to the following two possible definitions for the strict relations:

- The *standard method*, that amounts to define

$$[A <_\circ B] = \begin{cases} [A \leqslant_\circ B], & \text{if } [A \leqslant_\circ B] > [B \leqslant_\circ A] \\ 0, & \text{otherwise.} \end{cases}$$

  This means in fact to use (2) to define $[A <_\circ B]$ as the value of the strict order associated to the preorder $\leqslant_\circ$, where $\leqslant_\circ$ is either $\leqslant_{\forall\exists}$, $\leqslant_{\forall\exists 2}$ or $\leqslant_{\forall\forall}$.
- The *alternative method*, that first considers the strict order $<$ on companion of $\leqslant$ in W according to (2), and then defines $<_{\forall\exists}$, $<_{\forall\exists 2}$ and $<_{\forall\forall}$ on $\mathcal{P}(\mathrm{W})$ according to Definition 2, but replacing $\leqslant$ by $<$.

In general, these two methods give rise to *two different* irreflexive and (restricted) antisymmetric strict relations as the following examples show:

Example 1. Consider the $\forall\exists$ extension. Notice first that the alternative method gives

$$[A <_{\forall\exists} B] = \inf_{u \in A} \sup_{v \in B} [u < v].$$

The counterexample is the following. Take the four element set $W = \{u_1, u_2, u_3, u_4\}$, with the following fuzzy preorder: reflexivity ($[x \leqslant x] = 1$) plus

$[u_1 \leqslant u_3] = [u_3 \leqslant u_1] = a$ and $[u_2 \leqslant u_4] = b$, with $a, b \neq 0$. The associated strict relation on W is the one having only one pair of elements with value different from 0. Indeed an easy computation shows that $[u_2 < u_4] = b$. Let $A = \{u_1, u_2\}$ and let $B = \{u_3, u_4\}$. Then:

– To compute the value of $[A <_{\forall\exists} B]$ according to the standard method, we have to compute first:

  $[A \leqslant_{\forall\exists} B] = ([u_1 \leqslant u_3] \vee [u_1 \leqslant u_4]) \wedge ([u_2 \leqslant u_3] \vee [u_2 \leqslant u_4]) = a \wedge b \neq 0,$
  $[B \leqslant_{\forall\exists} A] = ([u_3 \leqslant u_1] \vee [u_3 \leqslant u_2]) \wedge ([u_4 \leqslant u_1] \vee [u_4 \leqslant u_2]) = a \wedge 0 = 0.$

  Then, by definition, we have $[A <_{\forall\exists} B] = a \wedge b \neq 0$.

– With the alternative method, the value of $[A <_{\forall\exists} B]$ is computed as

  $[A <_{\forall\exists} B] = \inf_{u \in A} \sup_{v \in B} [u < v] = 0.$

Therefore the obtained strict relations are different. □

Example 2. Consider now the $\forall\forall$ extension. Take $W = \{w_1, w_2\}$ with the preorder $[w_1 \leqslant w_1] = [w_2 \leqslant w_2] = [w_1 \leqslant w_2] = 1$ and $[w_2 \leqslant w_1] = 0$. Further, take $A = \{w_1\}$ and $B = W$. Then it is obvious that $[A \leqslant_{\forall\forall} B] = 1$ and $[B \leqslant_{\forall\forall} A] = 0$. Therefore, according to the standard method, we have $[A <_{\forall\forall} B] = 1$, while according to the alternative method, we have $[A <_{\forall\forall} B] = \inf_{u \in A} \inf_{v \in B} [u < v] = 0$. □

Notice that strict relations obtained by the alternative method are $\odot$-transitive (so they are strict orders), but this is not clear for strict relations obtained by the standard method. In fact we have the following open problems:

– Let $\leqslant$ be a strictly monotonic fuzzy preorder on W and let $\leqslant_\circ$ be one of the fuzzy preorders defined on $\mathcal{P}(W)$ considered in the previous sections. Is the strict relation obtained by the standard method $\odot$-transitive?
– Is there some order relation between the strict orders obtained by the standard and the alternative methods?
– It is obvious that the strict order $<$ on W and the strict order on $\mathcal{P}(W)$ obtained from the preorder by the standard method satisfy the following anti-symmetry property: for all $A, B \in \mathcal{P}(W)$, $\min([A <_\circ B], [B <_\circ A]) = 0)$. It is clear that for singletons the strict order obtained by the alternative method satisfies the same anti-symmetry property but, is this true for the strict order obtained by the alternative method in general? Otherwise, what type of anti-symmetry property does it satisfy?

Therefore, taking into account that we are interested in obtaining strict fuzzy orders (irreflexive and $\odot$-transitive relations), in the rest of the paper we will consider the strict relations obtained by the *alternative method* and its characteristics properties. Next theorem provides a characterization result for these strict orders.

Theorem 2. *The following characterizations hold:*

(i) *Let $\prec_{AE}$ be a relation between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 2, 3 and 4 of Prop. 3 plus irreflexivity ($[A \prec_{AE} A] = 0$) and restricted anti-symmetry ($\min([A \prec_{AE} B], [B \prec_{AE} A]) = 0$ for all singletons $A, B \in \mathcal{P}(\mathrm{W})$). Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\prec_{AE} = <_{\forall\exists}$.*

(ii) *Let $\prec_{AE2}$ be a relation between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 2, 3 and 4 of Prop. 4 plus irreflexivity and restricted anti-symmetry. Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\prec_{AE2} =<_{\exists\forall 2}$.*

(iii) *Let $\prec_{AA}$ be a relations between sets of $\mathcal{P}(\mathrm{W})$ satisfying Properties 2, 3, 4 and 5 of Prop. 5 plus irreflexivity and anti-symmetry. Then there exists a fuzzy $\odot$-preorder $\leqslant$ on the set W such that $\prec_{AA} =<_{\forall\forall}$.*

At the end of Section 2.2 we mentioned that one of the preorders we were interested in was the (crisp) relation $\leqslant_1$, whose fuzzy counterpart can be defined by $[x \leqslant_1 y] = \min([x \leqslant_{\forall\exists} y], [x \leqslant_{\forall\exists 2} y])$. Consequently, in Section 3 we separately characterized the fuzzy preorders $\leqslant_{\forall\exists}$ and $\leqslant_{\forall\exists 2}$, and the same is applicable to the corresponding strict orders studied in this section. We will move now to a logical approach to preference relations and to the previously studied fuzzy relations. In particular, in the next section we study a logical setting to reason about fuzzy preferences on classical propositions by means of several two-tiered modal logics, with binary modal operators corresponding to fuzzy preorders and strict orders separately, and after we show the desired preorder and strict order are definable in a yet another modal logic combining the previous ones.

5 A modal logic to reason about preferences

In this section three logics to reason about conditional (syntactic) objects capturing the idea of the preference relations $\leqslant_\circ$ (for $\circ \in \{\forall\exists, \forall\exists 2, \forall\forall\}$) are defined and studied, using similar techniques from [12].

Throughout this section, in order to simplify matters, rather than defining the logic relative to preference degrees in $[0, 1]$ and a t-norm, we will restrict ourselves to deal with a totally ordered finite set V of preference degrees (with 1 and 0 as its top and bottom elements), and we will fix a *finite t-norm* $\odot$ (see e.g. [19]) on V.

On these grounds, we define, model-theoretically, a common framework for several logics of preference relations, LAP for short, as follows.

Definition 3. *The language of* LAP *is two levelled:*

- *The first level ($\mathcal{L}_0$ language) contains propositional formulas of* LAP *that are built up from a finite set of variables $Var = \{p_1, \ldots, p_N\}$ and the constants $\bot, \top$ by means of the binary operators $\wedge$ and $\vee$ and the unary operator $\neg$. The set of propositional formulas is denoted by $\mathcal{P}$.*
- *The second level ($\mathcal{L}_1$ language) contains:*
  - *Atomic graded preference formulas of* LAP *that are triples*

$$\varphi \preceq^a \psi$$

consisting of two propositional formulas φ and ψ from $\mathcal{L}_0$, and a value $a \in V\backslash\{0\}$.
- *(General) preference formulas of* LAP *are built up from atomic graded preferences and the constants $\bot$, $\top$ by means of the binary connectives $\wedge$ and $\vee$ and the unary connective $\neg$.*

The semantics is given by $\odot$-preference Kripke models $\mathcal{M} = (W, \leqslant, e)$ where W is a finite set of worlds, $\leqslant$: W $\times$ W $\rightarrow V$ is a $\odot$-fuzzy preorder relation, and e : W $\times Var \mapsto \{0,1\}$ is a Boolean evaluation of propositional variables in every world, which is extended to propositions of $\mathcal{L}_0$ in the usual way for classical propositions. For each $\mathcal{L}_0$-proposition φ, we will denote by $[\varphi]_{\mathcal{M}}$ the set $\{w \in \mathrm{W} : e(w, \varphi) = 1\}$ of worlds satisfying φ.

For each $\circ \in \{\forall\exists, \forall\exists 2, \forall\forall\}$, each Kripke model $\mathcal{M} = (\mathrm{W}, S, e)$ induces a Boolean truth-evaluation of $\mathcal{L}_1$-formulas $e^{\circ}_{\mathcal{M}} : \mathcal{L}_1 \rightarrow \{0,1\}$ defined as follows:

– for atomic preference formulas: $e^{\circ}_{\mathcal{M}}(\varphi \preceq^a \psi) = 1$ if $[[\varphi]_{\mathcal{M}} \leqslant_{\circ} [\psi]_{\mathcal{M}}] \geqslant a$, and $e_{\mathcal{M}}(\varphi \preceq^a \psi) = 0$ otherwise.
– for compound formulas, use the usual Boolean truth functions.

From there, we can define the notion of logical consequence in the logic LAP for preference formulas.

Definition 4. *Let $\circ \in \{\forall\exists, \forall\exists 2, \forall\forall\}$. Let $T \cup \{\Phi\}$ be a set of preference formulas. We say that Φ logically follows from T under the $\leqslant_{\circ}$ semantics, written $T \models^{\circ}_{LAP} \Phi$, if for every Kripke model $\mathcal{M} = (\mathrm{W}, \leqslant, e)$, if $e^{\circ}_{\mathcal{M}}(\Psi) = 1$ for every $\Psi \in T$, then $e^{\circ}_{\mathcal{M}}(\Phi) = 1$ as well.*

In the following, for every Boolean evaluation ω of the propositional variables Var, we will denote by $\overline{\omega}$ the maximally elementary conjunction (m.e.c. for short) of all the N literals made true by ω. Obviously, every proposition is logically equivalent to a disjunction of m.e.c.'s.

Next subsections are devoted to the axiomatization of the particular logics for $\leqslant_{\forall\exists}, \leqslant_{\forall\exists 2}$ and $\leqslant_{\forall\forall}$.

5.1 The logic $\mathsf{LAP}_{\forall\exists}$ corresponding to the $\leqslant_{\forall\exists}$ preference relation

Recall that, when $\circ = \forall\exists$, the semantics we have in each Kripke model $\mathcal{M}$ is:

$$e_{\mathcal{M}}(\varphi \preceq^a \psi) = 1 \text{ iff } [[\varphi]_{\mathcal{M}} \leqslant_{\forall\exists} [\psi]_{\mathcal{M}}] = (\inf_{u \in [\varphi]_{\mathcal{M}}} \sup_{w \in [\psi]_{\mathcal{M}}} [u \leqslant w]) \geqslant a.$$

Building on this intended semantics, we propose the following axiomatization of $\mathsf{LAP}_{\forall\exists}$.

Definition 5. *The following are the axioms for $LAP_{\forall\exists}$:*

(A1) Axioms of classical propositional calculus (CPC) for $\mathcal{L}_1$-formulas
(A2) $\varphi \preceq^1 \psi$, where $\varphi \rightarrow \psi$ is a tautology of CPC

(A3) $(\varphi \preceq^a \psi) \wedge (\psi \preceq^b \chi) \to (\varphi \preceq^{a \odot b} \chi)$ *(transitivity)*

(A4) $(\varphi \preceq^a \psi) \to (\varphi \preceq^b \psi)$, *where* $a \leqslant b$ *(nestedness)*

(A5) $(\varphi \vee \psi \preceq^a \chi) \leftrightarrow (\varphi \preceq^a \chi) \wedge (\psi \preceq^a \chi)$ *(Left-OR)*

(A6) $(\overline{\omega} \preceq^a \varphi \vee \psi) \leftrightarrow (\overline{\omega} \preceq^a \varphi) \vee (\overline{\omega} \preceq^a \psi)$ *(restricted Right-OR)*

The only rule of $\mathsf{LAP}_{\forall\exists}$ *is Modus Ponens.*

We will denote by $\vdash^{\forall\exists}_{\mathsf{LAP}}$ the notion of deduction relative to the axiomatic system just defined.

Theorem 3. *For any set* $T \cup \{\Phi\}$ *of* $\mathcal{L}_1$*-formulas, it holds that* $T \models^{\forall\exists}_{\mathsf{LAP}} \Phi$ *if, and only if,* $T \vdash^{\forall\exists}_{\mathsf{LAP}} \Phi$*.*

Proof. One direction is soundness, and it is an easy computation, see Prop. 3. As for the other direction, assume $T \not\vdash^{\forall\exists}_{\mathsf{LAP}} \Phi$. The idea is to consider the graded expressions $\varphi \preceq^a \psi$ as propositional (Boolean) variables that are ruled by the axioms together with the laws of classical propositional logic CPC. Let Γ be the set of all possible instantiations of axioms (A1)-(A6). Then it holds that Φ does not follow from $T \cup \Gamma$ using CPC reasoning, i.e. $T \cup \Gamma \not\vdash_{CPC} \Phi$. By completeness of CPC, there exists a Boolean interpretation v such that $v(\Psi) = 1$ for all $\Psi \in T \cup \Gamma$ and $v(\Phi) = 0$. Now we will build a $\odot$-preference Kripke model $\mathcal{M}$ such that $e_{\mathcal{M}}(\Psi) = 1$ for all $\Psi \in T$ and $e_{\mathcal{M}}(\Phi) = 0$. To do that, we take $\Omega = \{\omega : Var \longrightarrow \{0,1\}\}$, i.e. the set of interpretations of propositional language, and define $\leqslant: \Omega \times \Omega \to V$ by

$$[\omega \leqslant \omega'] = \max\{a \in V \mid v(\overline{\omega} \preceq^a \overline{\omega'}) = 1\}.$$

By axioms (A2), (A3), $\leqslant$ is a $\odot$-preorder. Consider the model $\mathcal{M} = (\Omega, \leqslant, e)$, where for each $\omega \in \Omega$ and $p \in Var$, $e(\omega, p) = \omega(p)$. What remains to check is that $e_{\mathcal{M}}(\Psi) = v(\Psi)$ for every $\mathsf{LAP}_{\forall\exists}$-formula Ψ. In order to prove this equality it suffices to show that, for every $\varphi, \psi \in \mathcal{L}_0$ and $a \in [0,1]$, we have $e_{\mathcal{M}}(\varphi \preceq^a \psi) = v(\varphi \preceq^a \psi)$, that is, to prove that

$$v(\varphi \preceq^a \psi) = 1 \quad \text{iff} \quad \inf_{\omega \in [\varphi]_{\mathcal{M}}} \sup_{\omega' \in [\psi]_{\mathcal{M}}} [\omega \leqslant \omega'] \geqslant a.$$

By axioms (A5) and (A6), we have that $\mathsf{LAP}_{\forall\exists}$ proves

$$\varphi \preceq^a \psi \leftrightarrow \bigwedge_{\omega \in \Omega : \omega(\varphi) = 1} \bigvee_{\omega' \in \Omega : \omega'(\psi) = 1} \overline{\omega} \leqslant^a \overline{\omega'}.$$

Therefore, $v(\varphi \preceq^a \psi) = 1$ iff for all $\omega \in \Omega$ such that $\omega(\varphi) = 1$, there exists $w' \in \Omega$ such that $\omega'(\psi) = 1$ and $v(\overline{\omega} \preceq^a \overline{\omega'}) = 1$. But, as we have previously observed, $v(\overline{\omega} \preceq^a \overline{\omega'}) = 1$ holds iff $[\omega \leqslant \omega'] \geqslant a$. In other words, we actually have $v(\varphi \preceq^a \psi) = 1$ iff $\min_{\omega \in [\varphi]_{\mathcal{M}}} \max_{\omega' \in [\psi]_{\mathcal{M}}} [\omega \leqslant \omega'] \geqslant a$. This concludes the proof. □

5.2 The logics $\mathsf{LAP}_{\forall\exists 2}$ and $\mathsf{LAP}_{\forall\forall}$ corresponding to the $\leqslant_{\forall\exists 2}$ and $\leqslant_{\forall\forall}$ preference relations

In a very similar way, with the obvious changes, we can define axiomatic systems for the logics of $\mathsf{LAP}_{\forall\exists 2}$ and $\mathsf{LAP}_{\forall\forall}$. We do not include the completeness proofs since they are analogous to the one for $\mathsf{LAP}_{\forall\exists}$.

Recall that, under the $\forall\exists 2$ semantics, the evaluation of atomic preference formulas in a preference Kripke model $\mathcal{M}$ is as follows:

$$e_{\mathcal{M}}(\varphi \preceq^a \psi) = 1 \text{ iff } [[\varphi]_{\mathcal{M}} \leqslant_{\forall\exists 2} [\psi]_{\mathcal{M}}] = (\inf_{w \in [\psi]_{\mathcal{M}}} \sup_{u \in [\varphi]_{\mathcal{M}}} [u \leqslant w]) \geqslant a.$$

Theorem 4. *Let* $\mathsf{LAP}_{\forall\exists 2}$ *be the axiomatic system whose axioms are:*

(A1) *Axioms of CPC for* $\mathcal{L}_1$*-formulas*
(A2) $\varphi \preceq^1 \psi$, *where* $\psi \to \varphi$ *is a tautology of CPC*
(A3) $(\varphi \preceq^a \psi) \wedge (\psi \preceq^b \chi) \to (\varphi \preceq^{a \odot b} \chi)$ *(transitivity)*
(A4) $(\varphi \preceq^a \psi) \to (\varphi \preceq^b \psi)$, *for all* $a \leqslant b$ *(nestedness)*
(A5) $(\varphi \preceq^a \psi \vee \chi) \leftrightarrow (\varphi \preceq^a \psi) \wedge (\varphi \preceq^a \chi)$ *(Right-OR)*
(A6) $(\varphi \vee \psi \preceq^a \overline{\omega}) \leftrightarrow (\varphi \preceq^a \overline{\omega}) \vee (\preceq^a \psi \preceq^a \overline{\omega})$ *(restricted Left-OR)*

and whose only inference rule is modus ponens. Then $\mathsf{LAP}_{\forall\exists 2}$ *is sound and complete with respect to the class of* $\odot$*-preference Kripke models under the* $\forall\exists 2$ *semantics.*

As for the $\forall\forall$ semantics, the evaluation of atomic preference formulas in a preference Kripke model $\mathcal{M}$ is:

$$e_{\mathcal{M}}(\varphi \preceq^a \psi) = 1 \text{ iff } [[\varphi]_{\mathcal{M}} \leqslant_{\forall\forall} [\psi]_{\mathcal{M}}] = (\inf_{u \in [\varphi]_{\mathcal{M}}} \inf_{w \in [\psi]_{\mathcal{M}}} [u \leqslant w]) \geqslant a.$$

Theorem 5. *Let* $\mathsf{LAP}_{\forall\forall}$ *be the axiomatic system whose axioms are:*

(A1) *Axioms of CPC for* $\mathcal{L}_1$*-formulas*
(A2) $(\varphi \preceq^a \psi) \to (\varphi' \preceq^a \psi')$, *where* $\varphi' \to \varphi, \psi' \to \psi$ *are tautologies of CPC*
(A3) $\overline{\omega} \preceq^1 \overline{\omega}$ *(restricted reflexivity)*
(A4) $(\varphi \preceq^a \psi) \wedge (\psi \preceq^b \chi) \to (\varphi \preceq^{a \odot b} \chi)$ *(transitivity)*
(A5) $(\varphi \preceq^a \psi) \to (\varphi \preceq^b \psi)$, *for all* $a \leqslant b$ *(nestedness)*
(A6) $(\varphi \vee \psi \preceq^a \chi) \leftrightarrow (\varphi \preceq^a \chi) \wedge (\psi \preceq^a \chi)$ *(Left-OR)*
(A7) $(\psi \preceq^a \varphi \vee \psi) \leftrightarrow (\psi \preceq^a \varphi) \wedge (\psi \preceq^a \psi)$ *(Right-OR)*

and whose only inference rule is modus ponens. Then $\mathsf{LAP}_{\forall\forall}$ *is sound and complete with respect to the class of* $\odot$*-preference Kripke models under the* $\forall\forall$ *semantics.*

Moreover, in the same way, we could axiomatize the logics $\mathsf{LAP}^s_{\forall\exists}$, $\mathsf{LAP}^s_{\forall\exists 2}$ and $\mathsf{LAP}^s_{\forall\forall}$ corresponding to the associated strict preference orders.

Nevertheless our goal is to axiomatize the logic modeling preference triples $\langle \leqslant, <, \equiv \rangle$ corresponding to the preference relations $\leqslant_1 = \leqslant_{\forall\exists} \wedge \leqslant_{\forall\exists 2}$ and $\leqslant_2 = \leqslant_{\forall\forall}$. The axiomatizations of these logics are given in the next section.

5.3 The logic LAP^1

In this subsection we define and study the logic corresponding to the fuzzy preorder $\leqslant_1 = \leqslant_{\forall\exists} \wedge \leqslant_{\forall\exists 2}$.

The language of LAP^1 is as the one for LAP with the difference that now we have four kinds of atomic preference formulas:

$$\varphi \preceq^a_\alpha \psi, \quad \varphi \preceq^a_\beta \psi, \quad \varphi \prec^a_\alpha \psi, \quad \varphi \prec^a_\beta \psi,$$

where $a \in V\backslash\{0\}$. The semantics is still given by $\odot$-preference Kripke models $\mathcal{M} = (\mathrm{W}, \leqslant, e)$, where $e_\mathcal{M}$ evaluates the above kinds of atomic preference formulas in the expected way:

- $e_\mathcal{M}(\varphi \preceq^a_\alpha \psi) = 1$ if $[[\varphi]_\mathcal{M} \leqslant_{\forall\exists} [\psi]_\mathcal{M}] = (\inf_{u\in[\varphi]_\mathcal{M}} \sup_{w\in[\psi]_\mathcal{M}}[u \leqslant w]) \geqslant a$
- $e_\mathcal{M}(\varphi \preceq^a_\beta \psi) = 1$ if $[[\varphi]_\mathcal{M} \leqslant_{\forall\exists 2} [\psi]_\mathcal{M}] = (\inf_{w\in[\psi]_\mathcal{M}} \sup_{u\in[\varphi]_\mathcal{M}}[u \leqslant w]) \geqslant a$
- $e_\mathcal{M}(\varphi \prec^a_\alpha \psi) = 1$ if $[[\varphi]_\mathcal{M} <_{\forall\exists} [\psi]_\mathcal{M}] = (\inf_{u\in[\varphi]_\mathcal{M}} \sup_{w\in[\psi]_\mathcal{M}}[u < w]) \geqslant a$
- $e_\mathcal{M}(\varphi \prec^a_\beta \psi) = 1$ if $[[\varphi]_\mathcal{M} <_{\forall\exists 2} [\psi]_\mathcal{M}] = (\inf_{w\in[\psi]_\mathcal{M}} \sup_{u\in[\varphi]_\mathcal{M}}[u < w]) \geqslant a$.

The notion of logical consequence is defined as usual, and will be denoted by $\models_{\mathsf{LAP}^1}$.

Next we propose an axiomatic system for LAP^1.

Definition 6. *The axioms for* LAP^1 *are:*

- *Axioms of* $\mathsf{LAP}_{\forall\exists}$ *for the* $\preceq^a_\alpha$ *operators.*
- *Axioms of* $\mathsf{LAP}_{\forall\exists 2}$ *for the* $\preceq^a_\beta$ *operators.*
- *Axioms for the* $\prec^a_\alpha$ *operators:*

(AS1) $\neg(\varphi \prec^a_\alpha \varphi)$ *(irreflexivity)*
(AS2) $\neg((\overline{\varpi} \prec^a_\alpha \overline{\omega'}) \wedge (\overline{\omega'} \prec^b_\alpha \overline{\varpi}))$ *(restricted anti-symmetry)*
(AS3) $(\varphi \prec^a_\alpha \psi) \wedge (\psi \prec^b_\alpha \chi) \to (\varphi \prec^{a*b}_\alpha \chi)$ *($\odot$-transitivity)*
(AS4) $(\varphi \prec^a_\alpha \psi) \to (\varphi \prec^b_\alpha \psi)$*, for all* $a \leqslant b$ *(nestedness)*
(AS5) $(\varphi \prec^a_\alpha \overline{\varpi}) \wedge (\psi \prec^a_\alpha \overline{\varpi}) \leftrightarrow (\varphi \vee \psi \prec^a_\alpha \overline{\varpi})$ *(Restricted Left-OR)*
(AS6) $(\chi \prec^a_\alpha \varphi \vee \psi) \leftrightarrow (\chi \prec^a_\alpha \varphi) \vee (\chi \prec^a_\alpha \psi)$ *(Right-OR)*

- *Axioms for the* $\prec^a_\beta$ *operators:*

(BS1) $\neg(\varphi \prec^a_\beta \varphi)$ *(irreflexivity)*
(BS2) $\neg((\overline{\varpi} \prec^a_\beta \overline{\omega'}) \wedge (\overline{\omega'} \prec^b_\beta \overline{\varpi}))$ *(restricted anti-symmetry)*
(BS3) $(\varphi \prec^a_\beta \psi) \wedge (\psi \prec^b_\beta \chi) \to (\varphi \prec^{a*b}_\beta \chi)$ *($\odot$-transitivity)*
(BS4) $(\varphi \prec^a_\beta \psi) \to (\varphi \prec^b_\beta \psi)$*, for all* $a \leqslant b$ *(nestedness)*
(BS5) $(\varphi \vee \psi \prec^a_\beta \chi) \leftrightarrow (\varphi \prec^a_\beta \chi) \wedge (\psi \prec^a_\beta \chi)$ *(Left-OR)*
(BS6) $(\overline{\varpi} \prec^a_\beta \varphi \vee \psi) \leftrightarrow (\overline{\varpi} \prec^a_\beta \varphi) \vee (\overline{\varpi} \prec^a_\beta \psi)$ *(Restricted Right-OR)*

- *Connecting axioms:*

(AB) $\overline{\varpi} \preceq^a_\alpha \overline{\omega'} \leftrightarrow \overline{\varpi} \preceq^a_\beta \overline{\omega'}$

(ABS) $\overline{\omega} \prec_{\alpha}^{a} \overline{\omega'} \leftrightarrow \overline{\omega} \prec_{\beta}^{a} \overline{\omega'}$
(SA1) $\bigwedge \big((\overline{\omega} \preceq_{\alpha}^{a} \overline{\omega'}) \to (\overline{\omega'} \preceq_{\alpha}^{a} \overline{\omega}) : a > 0\big) \to \neg(\overline{\omega} \prec_{\alpha}^{a_0} \overline{\omega'})$,
where a_0 is the minimum element of $V \backslash \{0\}$.
(SA2) $\neg \bigwedge \big((\overline{\omega} \preceq_{\alpha}^{a} \overline{\omega'}) \to (\overline{\omega'} \preceq_{\alpha}^{a} \overline{\omega}) : a > 0\big) \to \big((\overline{\omega} \prec_{\alpha}^{b} \overline{\omega'}) \leftrightarrow (\overline{\omega} \preceq_{\alpha}^{b} \overline{\omega'})\big)$

The only inference rule for LAP^1 is Modus Ponens.

Observe that axiom (AB) is related to the fact that (semantically), over m.e.c.'s, the weak relations $\preceq_\alpha$ and $\preceq_\beta$ coincide, and the same for axiom (ABS) regarding the strict relations $\prec_\alpha$ and $\prec_\beta$. Finally, axioms (SA1) and (SA2) are for $\prec_\alpha$ a logical translation of the definition of strict order $<$ from the preorder $\leqslant$ on W according to Equation (2). Note that analogous axioms for $\prec_\beta$ can be derived using axiom (AB).

Denoting by $\vdash_{\mathsf{LAP}^1}$ the notion of proof in LAP^1, we have the following completeness result.

Theorem 6. *For any set $T \cup \{\Phi\}$ of $\mathcal{L}_1$-formulas, it holds that $T \models_{\mathsf{LAP}^1} \Phi$ if, and only if, $T \vdash_{\mathsf{LAP}^1} \Phi$.*

Proof. One direction is soundness. Let $M = (W, \leqslant, e)$ a $\odot$-preference Kripke model. Axiom (AB) holds in M since both preorders $\leqslant_{\forall\exists}$ and $\leqslant_{\forall\exists 2}$ are defined from the same preorder $\leqslant$ on W, and thus they coincide over the singletons. The same argument is valid for (ABS), exchanging preorder by strict order. Axioms (SA1) and (SA2) correspond to the definition of the strict order $<$ on W from the preorder $\leqslant$. The interpretation of (AS1) roughly says that, for elements of W, if $[u \leqslant v] \leqslant [v \leqslant u]$ then $[u < v] = 0$ and (AS2) says that if $[u \leqslant v] > [v \leqslant u]$ then $[u < v] = [u \leqslant v]$.

As for the converse direction, assume $T \not\vdash_{\mathsf{LAP}^1} \Phi$. The construction of the countermodel is very similar to that of Theorem 3, and the idea is again to consider all atomic preference formulas $\varphi \lhd^a \psi$ (with $\lhd \in \{\preceq_\alpha, \preceq_\beta, \prec_\alpha, \prec_\beta\}$) as propositional (Boolean) variables that are ruled by the laws of classical propositional logic CPC. Let Γ be the set of all possible instantiations of axioms of LAP^1. Then it follows that Φ does not follow from $T \cup \Gamma$ using CPC reasoning, i.e. $T \cup \Gamma \not\vdash_{CPC} \Phi$. By completeness of CPC, there exists a Boolean interpretation v such that $v(\Psi) = 1$ for all $\Psi \in T \cup \Gamma$ and $v(\Phi) = 0$. Now we will build a $\odot$-preference Kripke model $\mathcal{M} = (\Omega, \leqslant, e)$ such that $e_{\mathcal{M}}(\Psi) = 1$ for all $\Psi \in T$ and $e_{\mathcal{M}}(\Phi) = 0$. We take $\Omega = \{\omega : Var \longrightarrow \{0,1\}\}$, i.e. the set of Boolean interpretations of propositional variables, and define $\leqslant : \Omega \times \Omega \to [0,1]$ by

$$[\omega \leqslant \omega'] = \max\{a \in V \mid v(\overline{\omega} \preceq_{\alpha}^{a} \overline{\omega'}) = 1\}.$$

Notice that, by axiom(AB), this value is equal to $\max\{a \in V \mid v(\overline{\omega} \preceq_{\beta}^{a} \overline{\omega'}) = 1\}$. Based on $\leqslant$, we can define the corresponding strict order $<$, and from we can define the strict relations on subsets of W, $<_{\forall\exists}$ and $<_{\forall\exists 2}$, that coincide on the singletons by axiom (ABS). By the transitivity axioms of $\mathsf{LAP}_{\forall\exists}$ and $\mathsf{LAP}_{\forall\exists 2}$, $\leqslant$ is a $\odot$-preorder. We define now the evaluation function e, where for each $w \in \Omega$ and $p \in Var$, $e(w,p) = w(p)$. What remains to be checked is that $e_{\mathcal{M}}(\Psi) = v(\Psi)$

for every LAP^1-formula Ψ. In order to prove this equality it suffices to show that, for every $\varphi, \psi \in \mathcal{L}_0$ and $a \in V\backslash\{0\}$, we have $e_{\mathcal{M}}(\varphi \lhd^a \psi) = v(\varphi \lhd^a \psi)$. As mentioned above the proof is very similar to the one in Theorem 3 for all the atomic preference formulas, but specially when $\lhd \in \{\preceq_\alpha, \preceq_\beta\}$. Therefore we only prove the equality for atomic preference formulas of type $\varphi \prec^a_\beta \psi$. By the semantics of LAP^1,

$$e_{\mathcal{M}}(\varphi \prec^a_\beta \psi) = 1 \quad \text{iff} \quad \inf_{\omega' \in [\psi]_{\mathcal{M}}} \sup_{\omega \in [\varphi]_{\mathcal{M}}} [\omega < \omega'] \geqslant a.$$

By axioms (BS5) and (BS6), we have that LAP^1 proves

$$\varphi \prec^a_\beta \psi \;\leftrightarrow\; \bigwedge_{\omega' \in \Omega : \omega'(\psi)=1} \; \bigvee_{\omega \in \Omega : \omega(\varphi)=1} \overline{\omega} \leqslant^a \overline{\omega'}.$$

Therefore, $v(\varphi \prec^a_\beta \psi) = 1$ iff for all $\omega' \in \Omega$ such that $\omega'(\psi) = 1$, there exists $\omega \in \Omega$ such that $\omega(\varphi) = 1$ and $v(\overline{\omega} \prec^a_\beta \overline{\omega'}) = 1$. But $v(\overline{\omega} \prec^a_\beta \overline{\omega'}) = 1$ holds iff $[\omega < \omega'] \geqslant a$. In other words, we actually have $v(\varphi \prec^a_\beta \psi) = 1$ iff $\min_{\omega' \in [\psi]_{\mathcal{M}}} \max_{\omega \in [\varphi]_{\mathcal{M}}} [\omega \leqslant \omega'] \geqslant a$. This concludes the proof. □

In the logic LAP^1 we can define the following modal operators for the indifference relations corresponding to the preference modalities $\preceq^a_\alpha$ and $\preceq^a_\alpha$:

- $\varphi \equiv^a_\alpha \psi$ as $(\varphi \preceq^a_\alpha \psi) \wedge (\psi \preceq^a_\alpha \varphi)$,
- $\varphi \equiv^a_\beta \psi$ as $(\varphi \preceq^a_\beta \psi) \wedge (\psi \preceq^a_\beta \varphi)$,

and, from them, we can in turn define the modalities

- $\varphi \preceq^a_1 \psi$ as $(\varphi \preceq^a_\alpha \psi) \wedge (\varphi \preceq^a_\beta \psi)$,
- $\varphi \equiv^a_1 \psi$ as $(\varphi \equiv^a_\alpha \psi) \wedge (\varphi \equiv^a_\beta \psi)$,
- $\varphi \prec^a_1 \psi$ as $((\varphi \equiv^a_\alpha \psi) \wedge (\varphi \prec^a_\beta \psi)) \vee$
  $((\varphi \equiv^a_\beta \psi) \wedge (\varphi \prec^a_\alpha \psi)) \vee ((\varphi \prec^a_\alpha \psi) \wedge (\varphi \prec^a_\beta \psi))$.

that eventually determine $\langle \preceq_1, \equiv_1, \prec_1 \rangle$ as the preference structure of the logic LAP^1.

We finish this section with one remark justifying the above definition of $\prec^a_1$. Observe that given a preorder $\leqslant$ on W, the preorder $\leqslant_1$ on $\mathcal{P}(\mathrm{W})$ satisfies the following equation:

$$[A \leqslant_1 B] = \min([A \leqslant_{\forall\exists} B], [A \leqslant_{\forall\exists 2} B]),$$

that, by Equation (1), is equal to

$$\min(\max([A \equiv_{\forall\exists} B], [A <_{\forall\exists} B]), \max([A \equiv_{\forall\exists 2} B], [A <_{\forall\exists 2} B])),$$

and hence, also equal to

$$\begin{aligned}\max(\, & \min([A \equiv_{\forall\exists} B], [A \equiv_{\forall\exists 2} B]), \min([A \equiv_{\forall\exists} B], [A <_{\forall\exists 2} B]),\\ & \min([A <_{\forall\exists} B], [A \equiv_{\forall\exists 2} B]), \min([A <_{\forall\exists} B], [A <_{\forall\exists 2} B])\,).\end{aligned}$$

Thus, once we define $[A \equiv_1 B] = \min([A \equiv_{\forall\exists} B], [A \equiv_{\forall\exists 2} B])i$, then, again according to Equation (1), it seems very reasonable to define the strict order value $[A <_1 B]$ by the maximum of the three remaining terms above, that is:

$$[A <_1 B] = \max(\ \min([A \equiv_{\forall\exists} B], [A <_{\forall\exists 2} B]), \\ \min([A <_{\forall\exists} B], [A \equiv_{\forall\exists 2} B]), \min([A <_{\forall\exists} B], [A <_{\forall\exists 2} B])\ ).$$

This motivates the definition of $\varphi \prec_1^a \psi$ above.

5.4 The logic LAP^2

In this subsection we define and study the logic corresponding to the fuzzy preorder $\leqslant_2 = \leqslant_{\forall\forall}$.

The logic LAP^2 is defined as the expansion of $\mathsf{LAP}_{\forall\forall}$ with modal operators for the strict preference $\prec^a$, for each $a \in V\backslash\{0\}$. We just need to take into account that the semantics for the $\prec^a$ operators is as expected: given a Kripke model $\mathcal{M} = (\mathrm{W}, \leqslant, e)$,

– $e_{\mathcal{M}}(\varphi \prec^a \psi) = 1$ if $\quad [[\varphi]_{\mathcal{M}} <_{\forall\forall} [\psi]_{\mathcal{M}}] = (\inf_{u \in [\varphi]_{\mathcal{M}}} \inf_{w \in [\psi]_{\mathcal{M}}} [u < w]) \geqslant a$.

Definition 7. *The axioms for* LAP^2 *are the ones of* $\mathsf{LAP}_{\forall\forall}$ *for the* $\preceq^a$ *operators plus:*

(AS1) $(\varphi \prec^a \psi) \to (\varphi' \prec^a \psi')$, *where* $\varphi' \to \varphi, \psi' \to \psi$ *are tautologies of CPC*

(AS2) $\neg(\varphi \prec^a \varphi)$ *(irreflexivity)*

(AS3) $(\varphi \prec^a \psi) \wedge (\psi \prec^b \chi) \to (\varphi \prec^{a \odot b} \chi)$ *($\odot$-transitivity)*

(AS4) $(\varphi \prec^a \psi) \to (\varphi \prec^b \psi)$, *for all* $a \leqslant b$ *(nestedness)*

(AS5) $(\varphi \vee \psi \prec^a \chi) \ \leftrightarrow\ (\varphi \prec^a \chi) \wedge (\psi \prec^a \chi)$ *(Left-OR)*

(AS6) $(\psi \prec^a \varphi \vee \chi) \ \leftrightarrow\ (\psi \prec^a \chi) \wedge (\psi \prec^a \chi)$ *(Right-OR)*

(SA1) $\bigwedge \big((\overline{\omega} \preceq^a \overline{\omega'}) \to (\overline{\omega'} \preceq^a \overline{\omega}) : a > 0\big) \to \neg(\overline{\omega} \prec^{a_0} \overline{\omega'})$, *where* a_0 *is the minimum element of* $V\backslash\{0\}$.

(SA2) $\neg \bigwedge \big((\overline{\omega} \preceq^a \overline{\omega'}) \to (\overline{\omega'} \preceq^a \overline{\omega}) : a > 0\big) \to \big((\overline{\omega} \prec^b \overline{\omega'}) \ \leftrightarrow\ (\overline{\omega} \preceq^b \overline{\omega'})\big)$

The only rule of LAP^2 *is modus ponens.*

Note that axioms (SA1) and (SA2) above are analogous to the ones in LAP^1, and the remark after the definition LAP^1 justifying them applies here as well.

The completeness theorem is ready and the proof is also analogous to previous ones, thus we omit it.

Theorem 7. *For any set* $T \cup \{\Phi\}$ *of* $\mathcal{L}_1$*-formulas, it holds that* $T \models_{LAP^1} \Phi$ *if, and only if,* $T \vdash_{\mathsf{LAP}^1} \Phi$.

Finally, let us observe that in LAP^2 we can also define now the preference structure $\langle \preceq_2, \equiv_2, \prec_2 \rangle$ in the obvious way:

- The weak preference statement $\varphi \preceq_2 \psi$ is defined as $\varphi \preceq^a \psi$,
- The equivalence statement $\varphi \equiv_2 \psi$ is defined as $(\varphi \preceq^a \psi) \wedge (\psi \preceq^a \varphi)$,
- The strict preference statement $\varphi \prec_2 \psi$ is defined by $(\varphi \prec^a \psi)$.

Notice however that, strictly speaking, $\varphi \preceq_2 \psi$ is not a fuzzy preorder and $\equiv_2$ is not a fuzzy similarity since they are not reflexive.

6 Conclusions and future work

In this paper we have studied preference structures on classical sets arising from fuzzy preference relations, a topic that, as far as we know, has not been very studied in the literature. We have approached the question both from a relational and logical points of view. In the relational approach we have studied and characterized possible extensions of fuzzy preorders on a crisp set W (interpreted as fuzzy preferences between the elements of W) to crisp subsets of W (fuzzy preferences on crisps subsets). Within the logical approach, we have defined and studied several two-tiered modal logics capturing, at the syntactical level, the corresponding preference structures. The same scheme can be generalized to fuzzy preference relations on fuzzy sets. Given a fuzzy preorder $\leqslant$ on a universe W, we can define corresponding extensions to fuzzy relations on the set $\mathcal{F}(\mathrm{W})$ of fuzzy subsets of W. For example, for all $A, B \in \mathcal{F}(\mathrm{W})$, corresponding extensions for $\forall\exists$ and $\forall\forall$ could be defined as

$$(A \leqslant_{\forall\exists} B) = [\inf_{u \in W}((\mu_A(u)) \to (\sup_{v \in W}([u \leqslant v] \odot \mu_B(v))))]$$
$$(A \leqslant_{\forall\forall} B) = [\inf_{u \in W}((\mu_A(u)) \to (\inf_{v \in W}([u \leqslant v] \to \mu_B(v))))].$$

As future work we plan to study and characterize these type of fuzzy preference relations taking into account the works by Bodenhofer et al. [2–4], where the authors study some of these relations in the purely fuzzy relational setting. Finally we plan to connect the corresponding fuzzy preference structures with a modal many-valued logic framework, with necessity, possibility, universal and existential modal operators (see [21] for a first approach) in a similar way that it is done in [1] in the classical setting.

Acknowledgments Esteva and Godo acknowledges partial support by the Spanish FEDER/MINECO project TIN2015-71799-C2-1-P. Vidal acknowledges partial support by the joint project of the Austrian Science Fund (FWF) I1897-N25 and the Czech Science Foundation (GACR) 15-34650L, and by the institutional grant RVO:67985807.

Dedication This paper is our humble contribution to the tribute, in the occasion of his 65th birthday, to José Luis “Curro” Verdegay. Excellent researcher and better person, he has been one of the pioneers of fuzzy logic in Spain and founder and driving force of the research group on Decision Making and Optimization at the University of Granada. Our contribution is devoted to logic and fuzzy preferences, a topic that, although it is not central on the research of Curro, is ubiquitous in fuzzy decision making models and we hope it may be of his interest. Along many years, we have jointly participated in many events around the world with Curro and with our friends from Granada, we have learnt a lot from his research ideas and organizational competences, but more importantly, we have enjoyed his friendship and shared many unforgettable moments. Thanks for all Curro, and congratulations for this well-deserved homage!

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