On finite-valued bimodal logics with an application to reasoning about preferences

Amanda Vidal[1], Francesc Esteva[2], and Lluis Godo[2]

[1] Institute of Computer Science (ICS - CAS),
Pod vodárenskou věží 2, 182 07 Prague, Czech Republic,
amanda@cs.cas.cz
[2] Artificial Intelligence Research Institute (IIIA - CSIC),
Campus de la UAB s/n, 08193 Bellaterra, Spain,
{esteva,godo}@iiia.csic.es

Abstract. In a previous paper by Bou et al., the minimal modal logic over a finite residuated lattice with a necessity operator $\Box$ was characterized under different semantics. In the general context of a residuated lattice, the residual negation $\neg$ is not necessarily involutive, and hence a corresponding possibility operator cannot be introduced by duality. In the first part of this paper we address the problem of extending such a minimal modal logic with a suitable possibility operator $\Diamond$. In the second part of the paper, we introduce suitable axiomatic extensions of the resulting bimodal logic and define a logic to reason about fuzzy preferences, generalising to the many-valued case a basic preference modal logic considered by van Benthem et al.

Keywords: many-valued modal logic, necessity and possibility modal operators, finite residuated lattice, reasoning about graded preferences

1 Introduction

Theoretical studies of fuzzy or many-valued modal logics have attracted an increasing attention in the last years, both following general and foundational approaches e.g. in [15, 4, 9, 14], as well as focusing on particular families of fuzzy logics, mainly those based on Gödel logic [7, 6, 8, 11, 10], Łukasiewicz logic [12, 3] or more recently on Product logic [17].

In particular, in [4] the authors study in depth minimal modal logics with a necessity operator $\Box$ (and canonical truth-constants) over a finite residuated lattice, considering different classes of many-valued Kripke frames and getting complete axiomatizations with respect to them.

In the first part of this paper, Section 2, we address the problem of extending those minimal modal logics with a possibility modal operator $\Diamond$. Note that in the general context of a residuated lattice, if the residual negation $\neg$ is not involutive, then $\Box$ and $\Diamond$ are not dual in the usual sense ($\Diamond$ is not definable as $\neg\Box\neg$).

In the second part, in Section 3 we define suitable axiomatic extensions of the above fuzzy bi-modal logics, and then in Section 4 we define a logic to reason about fuzzy preferences, generalising to the many-valued case one of the preference modal logics considered by van Benthem et al. in [1].

2 The minimal bimodal logic of a finite residuated lattice

We start from basic definitions in [4], with which we assume the reader certain familiarity. Through the following sections, we will be assuming $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ to denote a *finite* (bounded, integral, commutative) residuated lattice, and we will consider its canonical expansion $\mathbf{A^c}$ by adding a new constant $\overline{a}$ for every element $a \in A$ (canonical in the sense that the interpretation of $\overline{a}$ in $\mathbf{A^c}$ is a itself.) The logic associated with $\mathbf{A^c}$ will be denoted by $\boldsymbol{\Lambda}(\mathbf{A^c})$, and its logical consequence relation $\models_{\mathbf{A^c}}$ is defined as follows: for all sets $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ of formulas built in the usual way from a set of propositional variables $\mathcal{V}$ in the language of residuated lattices (possibly including constants from $\{\overline{a} : a \in A\}$),

$$\Gamma \models_{\mathbf{A^c}} \phi \quad \iff \quad \forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c),\ \text{if } h[\Gamma] \subseteq \{1\} \text{ then } h(\phi) = 1, \tag{1}$$

where $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c)$ denotes the set of evaluations of formulas on $\mathbf{A^c}$.

In order to introduce the minimum bimodal logic over $\mathbf{A^c}$, let us consider the modal language $\mathbf{MFm}$ being the expansion of $\mathbf{Fm}$ with two modal operators $\Box$ and $\Diamond$. Kripke-style semantics for the bimodal logic is defined as follows.

Definition 1. *An* $\mathbf{A}$-Kripke model *is a triple* $\mathfrak{M} = \langle W, R, e\rangle$ *where*
- W *is a set of worlds,*
- $R\colon W \times W \to A$, *is an A-valued accessibility relation between worlds,*
- $e\colon W \times \mathcal{V} \to A$ *is the evaluation of the model, and it is uniquely extended to formulas by letting* $e(w, \overline{a}) = a$ *for every* $a \in A$, $e(w, \varphi\star\psi) \coloneqq e(w, \varphi)\star e(w, \psi)$ *for any propositional connective* $\star$ *in the language,*[3] *and*

$$e(w, \Box\varphi) \coloneqq \bigwedge_{w\in W} \{R(v, w) \to e(w, \varphi)\}, \quad e(w, \Diamond\varphi) \coloneqq \bigvee_{w\in W} \{R(v, w)\& e(w, \varphi)\}.$$

We let $\mathbb{BM}_{\mathbf{A}}$ *denote the class of all* $\mathbf{A}$*-Kripke models.*

Observe that the above values are always well-defined because the lattice is finite.

We say that, in a Kripke model $\mathfrak{M}$, a formula φ follows from a set of premises Γ, and write $\Gamma \Vdash_{\mathfrak{M}} \varphi$, whenever for any $v \in W$ such that $e(v, \gamma) = 1$ for all $\gamma \in \Gamma$, it holds that $e(v, \varphi) = 1$ too. Whenever we have a class of models $\mathbb{C}$, we will write $\Gamma \Vdash_{\mathbb{C}} \varphi$ meaning that φ follows from Γ in all the models of the class.[4]

As usual, in any deductive system used along this article (including the ones defined in the above lines), we will omit writing $\emptyset$ whenever the set of premises is empty, and simply write $\vdash \varphi$.

[3] For the sake of clarity, we use the same symbol (e.g. $\odot, \rightarrow$) both as syntactic connective in the language $\mathbf{MFm}$ and as the corresponding algebraic operation.

[4] This logical consequence is usually referred to as the *local* modal logic arising from a class of Kripke models, in contrast with the *global* one that considers truth in the whole model. It is out of the scope of this work to introduce and study the global modal logic over residuated lattices.

2.1 Axiomatization

In this section we present a modal axiomatic system over a finite residuated lattice $\mathbf{A}$, as an extension of the axiomatic modal system presented in [4, Def. 4.6], called $\boldsymbol{\Lambda}(\mathsf{Fr}, \mathbf{A^c})$, and that will be shown to be complete with respect to $\Vdash_{\mathbb{BM}_\mathbf{A}}$ defined above.

Before proceeding to the definition of the axiomatic system, observe that a sort of symmetric version of the axiom (Ax_a) in [4, Prop. 3.10] is valid in all $\mathbf{A}$-Kripke models. Namely, for every $a \in A$,

$$\Vdash_{\mathbb{BM}_\mathbf{A}} \Box(\varphi \to \overline{a}) \leftrightarrow (\Diamond\varphi \to \overline{a}).$$

This follows immediately from the fact that in any residuated lattice $\mathbf{A}$, for any $X \cup \{a\} \subseteq A$, it holds that $\bigwedge_{x \in X} \{x \to a\} = \bigvee_{x \in X} \{x\} \to a$ whenever the corresponding inf. and sup. exist (which is our case since the algebra is finite).

It looks then natural to consider that formula as a member of the axiomatic system, and as we prove below, this one is indeed the only formula referring to $\Diamond$ that we need to consider in order to get a complete axiomatic system for $\Vdash_{\mathbb{BM}_\mathbf{A}}$.

Definition 2. *Let $\mathsf{BM}_\mathbf{A}$ be the deductive system given by:*

1. *The axiomatic basis of $\boldsymbol{\Lambda}(\mathsf{Fr}, \mathbf{A^c})$, i.e.:*
   - *an axiomatic basis for $\boldsymbol{\Lambda}(\mathbf{A^c})$*
   - *modal axioms for $\Box$:*
     $\Box\overline{1}$, (MD) $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$, (Ax_a) $\Box(\overline{a} \to \varphi) \leftrightarrow (\overline{a} \to \Box\varphi)$
2. *The axiom schemata*
   $(\Box\Diamond_a)$ $\Box(\varphi \to \overline{a}) \leftrightarrow (\Diamond\varphi \to \overline{a})$, *for each* $a \in A$
3. *The rules of the basis for $\boldsymbol{\Lambda}(\mathbf{A^c})$ and the Monotonicity rule:*
   (Mon) *from* $\varphi \to \psi$ *derive* $\Box\varphi \to \Box\psi$

We will denote by $\vdash_{\mathsf{BM}_\mathbf{A}}$ the corresponding notion of proof, and by $Th(\mathsf{BM}_\mathbf{A})$ the set of theorems of the logic $\mathsf{BM}_\mathbf{A}$, i.e. $Th(\mathsf{BM}_\mathbf{A}) = \{\varphi \in \mathbf{MFm} : \vdash_{\mathsf{BM}_\mathbf{A}} \varphi\}$.

In order to prove completeness of the previous logic with respect to the relation $\Vdash_{\mathbb{BM}_\mathbf{A}}$ we will resort to the usual canonical model construction. However, we need to define a canonical model different from the one used in [4, Lemma 4.8] in order to capture the behaviour of the $\Diamond$ operator. Before doing so, let us state a useful lemma that will allow to move from deductions in the modal logic $\mathsf{BM}_\mathbf{A}$ to deductions in the underlying propositional logic $\boldsymbol{\Lambda}(\mathbf{A^c})$.

Lemma 1. *For any $\Gamma \cup \{\varphi\} \subseteq MFm$, $\Gamma \vdash_{\mathsf{BM}_\mathbf{A}} \varphi$ iff $Th(\mathsf{BM}_\mathbf{A}) \cup \Gamma \vdash_{\boldsymbol{\Lambda}(\mathbf{A})} \varphi$.*[5]

Proof. Right-to-left direction is immediate, since $\mathsf{BM}_\mathbf{A}$ expands $\boldsymbol{\Lambda}(\mathbf{A^c})$. The other direction is easily proved by induction on the length of the proof of φ from Γ, observing that the rule (Mon), the only new inference rule added to $\boldsymbol{\Lambda}(\mathbf{A^c})$ in doing the modal expansion, only applies to theorems of the logic. □

[5] We do not detail this issue here due to lack of space and interest, but for the interested reader, it should be clear that the language from the right side of this equivalence counts with an extended -countable- set of variables that capture the modal formulas.

If $\Gamma \not\vdash_{\mathsf{BM}_{\mathbf{A}}} \varphi$, the previous result allows us to obtain a non-modal homomorphism h that evaluates to 1 the formulas in Γ and all theorems of $\mathsf{BM}_{\mathbf{A}}$, and does not do the same for φ. This is the reason behind the definition of the canonical model that follows.

Definition 3. *The* ***canonical Kripke model*** *of* $\mathsf{BM}_{\mathbf{A}}$ *is the* $\mathbf{A}$*-valued model* $\mathfrak{M}^c = \langle W^c, R^c, e^c\rangle$ *where:*

- $W^c := \{h \in Hom(\mathbf{MFm}, \mathbf{A})\colon h(Th(\mathsf{BM}_{\mathbf{A}})) = \{1\}\}$,
- $R^c vw := \bigwedge_{\psi\in Fm} (v(\Box\psi) \to w(\psi)) \wedge (w(\psi) \to v(\Diamond\psi))$,
- $e^c(v, p) = v(p)$*, for any propositional variable* p*.*

As usual, the key fact in using the previously defined model to prove completeness is that it enjoys the so-called *truth lemma*, ensuring that the behaviour of e^c coherently extends to all formulas.

Lemma 2 (Truth lemma). *For any* $v \in W^c$ *and any modal formula* φ*, it holds that* $e^c(v, \varphi) = v(\varphi)$*.*

The previous lemma can be proved by structural induction, the only non trivial cases being the formulas beginning by a modality. One of the inequalities (for both modalities) is easy to prove, as shown next.

Lemma 3. *For any formula* $\varphi \in Fm$*, and any* $v \in W^c$*, the following hold:*

1. $v(\Diamond\varphi) \geqslant \bigvee_{w\in W^c} \{R^c(v, w) \odot w(\varphi)\}$,
2. $v(\Box\varphi) \leqslant \bigwedge_{w\in W^c} \{R^c(v, w) \to w(\varphi)\}$.

Proof. We prove the first inequality, the other can be proved analogously. Applying the definition of $R^c(v, w)$ and the monotonicity of $\odot$ in any residuated lattice, it is possible to prove the following inequality for any $v, w \in W^c$:

$$\begin{aligned} R^c(v, w) \odot w(\varphi) &= \bigwedge_{\psi\in Fm} (v(\Box\psi) \to w(\psi)) \wedge (w(\psi) \to v(\Diamond\psi)) \odot w(\varphi) \\ &\leqslant (w(\varphi) \to v(\Diamond\varphi)) \odot w(\varphi) \leqslant v(\Diamond\varphi). \end{aligned}$$

Since this holds for any world w, we have $\bigvee_{w\in W^c} \{R^c(v, w) \odot w(\varphi)\} \leqslant v(\Diamond\varphi)$. □

As for the converse inequalities, it is worth to first prove a powerful technical lemma (cf. [16, Lem. 6.12]) that generalizes and provides a more modular and scalable proof of the truth lemma compared to that in [4, Lem. 4.8]

Lemma 4. *Let* $v \in W^c$ *and* $\varphi \in Fm$ *be such that for all* $w \in W^c$ *it holds that* $R^c(v, w) \leqslant w(\varphi)$*. Then* $v(\Box\varphi) = 1$*.*

Proof. For the sake of a clearer notation, let $\sigma\colon Fm \to Fm$ be given by

$$\sigma(\psi) = (\overline{v(\Box\psi)} \to \psi) \wedge (\psi \to \overline{v(\Diamond\psi)}).$$

In this way, we have that $R^c vw = \bigwedge_{\psi \in Fm} w(\sigma(\psi))$. Now, observe that by definition, $R^c(v,w) \leqslant w(\varphi)$ if and only if

$$\text{for all } a \in A, \text{ if } a \leqslant R^c vw \text{ then } a \leqslant w(\varphi). \tag{2}$$

By hypothesis, this holds for each $w \in W^c$. Unfolding all the definitions, this means that for any $w \in Hom(\mathbf{Fm}, \mathbf{A})$ such that $w(Th(\mathsf{BM}_{\mathbf{A}})) = \{1\}$, and for any $a \in A$, if $a \leqslant w(\sigma(\psi))$ for all $\psi \in Fm$ then $a \leqslant w(\varphi)$. Clearly, we can now formulate this fact in terms of the propositional consequence relation $\models_{\mathbf{A}}$:

$$Th(\mathsf{BM}_{\mathbf{A}}) \cup \{\overline{a} \to \sigma(\psi) \colon \psi \in Fm\} \models_{\mathbf{A}^c} \overline{a} \to \varphi. \tag{3}$$

Since the propositional logic is finitary, then for each $a \in A$ there is a finite set of formulas Σ_a,[6] such that (3) holds iff

$$Th(\mathsf{BM}_{\mathbf{A}}) \cup \{\overline{a} \to \bigwedge_{\psi \in \Sigma_a} \sigma(\psi)\} \models_{\mathbf{A}^c} \overline{a} \to \varphi.$$

Let $\Sigma \coloneqq \bigcup_{a \in A} \Sigma_a$, which is clearly finite. Since $\models_{\mathbf{A}^c} \bigwedge\limits_{\psi \in \Sigma} \sigma(\psi) \to \bigwedge\limits_{\psi \in \Sigma_a} \sigma(\psi)$, we have for each $a \in A$, $Th(\mathsf{BM}_{\mathbf{A}}) \cup \{\overline{a} \to \bigwedge\limits_{\psi \in \Sigma} \sigma(\psi)\} \models_{\mathbf{A}^c} \overline{a} \to \varphi$, from where

$$Th(\mathsf{BM}_{\mathbf{A}}) \models_{\mathbf{A}^c} \bigwedge_{\psi \in \Sigma} \sigma(\psi)) \to \varphi,$$

by taking for each $h \in Hom(\mathbf{Fm}, \mathbf{A})$ such that $h(Th(\mathsf{BM}_{\mathbf{A}}),) = 1$, the constant $a = h(\bigwedge_{\psi \in \Sigma} \sigma(\psi))$ in the deduction above.

We can now successively apply completeness of $\Lambda(\mathbf{A})$ w.r.t $\models_{\mathbf{A}}$, Lemma 1, (Mon) rule and then Lemma 1 and non-modal completeness again to get

$$Th(\mathsf{BM}_{\mathbf{A}}) \models_{\mathbf{A}^c} \Box(\bigwedge_{\psi \in \Sigma} \sigma(\psi)) \to \Box\varphi.$$

To conclude the proof it suffices to check that $v(\Box(\bigwedge_{\psi \in \Sigma} \sigma(\psi)) = 1$. By axiom (MD), $v(\Box(\bigwedge_{\psi \in \Sigma} \sigma(\psi)) = v(\bigwedge_{\psi \in \Sigma} \Box\sigma(\psi))$, so we only need to check $v(\Box\sigma(\psi)) = 1$ for each $\sigma \in \Sigma$. This is proved by the following chain of equalities:

$$\begin{aligned} v(\Box\sigma(\psi)) =& v(\Box((\overline{v(\Box\psi)} \to \psi) \wedge (\psi \to \overline{v(\Diamond\psi)}))) \\ =& v(\Box(\overline{v(\Box\psi)} \to \psi) \wedge \Box(\psi \to \overline{v(\Diamond\psi)})) \\ =& v((\overline{v(\Box\psi)} \to \Box\varphi) \wedge (\Diamond\varphi \to \overline{v(\Diamond\psi)})) \\ =& (\overline{v(\Box\psi)} \to v(\Box\varphi)) \wedge (v(\Diamond\varphi) \to \overline{v(\Diamond\psi)}) = 1. \end{aligned}$$ □

We have now the two main pieces to provide a clear proof of the truth lemma.

[6] Note that for a finite set of formulas Θ, $\bigwedge\limits_{\theta \in \Theta} \theta$ is a formula in the language too.

Proof of Lemma 2. Let us prove the converse inequalities of Lemma 3. Let $\varphi = \Box\psi$ for some ψ. Since we already know that $e(v, \Box\psi) \geqslant v(\Box\psi)$, to prove the equality is enough to prove that for all $a \in A$, if $a \leqslant e(v, \Box\psi)$ then $a \leqslant v(\Box\psi)$. Thus, let $a \in A$ be such that

$$a \leqslant e(v, \Box\psi) = \inf\{R^c(v, w) \to e(w, \psi) : w \in W^c\}.$$

By the induction hypothesis, it is enough to prove that $a \leqslant R^c(v, w) \to w(\psi)$ for all $w \in W$. By residuation, $R^c(v, w) \leqslant a \to w(\psi)$, and so,

$$R^c(v, w) \leqslant w(\overline{a} \to \psi) \quad \text{for all } w \in W^c.$$

Lemma 4 implies that $v(\Box(\overline{a} \to \psi)) = 1$. Then, by axiom (Ax_a) we get that $a \to v(\Box\psi) = 1$, and so, $a \leqslant v(\Box\psi)$.

In a very similar way we can prove the analogous result for $\Diamond$. Let $\varphi = \Diamond\psi$ for some ψ. To check that $e(v, \Diamond\psi) \geqslant v(\Diamond\psi)$ is enough to prove that for all $a \in A$, if $a \geqslant e(v, \Diamond\psi)$ then $a \geqslant v(\Diamond\psi)$. Thus, let $a \in A$ be such that

$$a \geqslant e(v, \Diamond\psi) = \sup\{R^c(v, w) \odot e(w, \psi) : w \in W^c\}.$$

By induction, this is equivalent to $a \geqslant R^c(v, w) \odot w(\psi)$ for all $w \in W$. By residuation, $R^c(v, w) \leqslant w(\psi) \to a$, and so,

$$R^c(v, w) \leqslant w(\psi \to \overline{a}) \quad \text{for all } w \in W^c.$$

From Lemma 4, we know that $v(\Box(\psi \to \overline{a})) = 1$. Then, by axiom $(\Box\Diamond_a)$ we get that $v(\Diamond\psi) \to a = 1$, and so, $a \geqslant v(\Diamond\psi)$. □

Completeness of $\mathsf{BM}_{\mathbf{A}}$ is now a corollary of the Truth Lemma and Lemma 1.

Theorem 1 (Completeness of $\mathsf{BM}_{\mathbf{A}}$). *For any $\Gamma \cup \{\varphi\} \subseteq \mathbf{MFm}$, $\Gamma \vdash_{\mathsf{BM}_{\mathbf{A}}} \varphi$ iff $\Gamma \Vdash_{\mathbb{BM}_{\mathbf{A}}} \varphi$.*

Proof. Soundness was already justified before Definition 2. Concerning completeness, let $\Gamma \nvdash_{\mathsf{BM}_{\mathbf{A}}} \varphi$. From Lemma 1, there is a homomorphism h from MFm into A evaluating to 1 all theorems of $\mathsf{BM}_{\mathbf{A}}$ and all elements in Γ, and such that $h(\varphi) < 1$. Then, h is by definition a world of the canonical model of $\mathsf{BM}_{\mathbf{A}}$. Using the truth lemma, we know that $e^c(h, \gamma) = h(\gamma) = 1$ for all $\gamma \in \Gamma$, while $e^c(h, \varphi) = h(\varphi) < 1$, and so, the canonical model serves to prove that $\Gamma \nVdash_{\mathbb{BM}_{\mathbf{A}}} \varphi$.

3 Some useful axiomatic extensions

It is reasonable to ask ourselves whether some interesting frame and model conditions can be characterized by means of adding some axiom schemata to the system $\mathsf{BM}_{\mathbf{A}}$. While a systematic study of these properties is far from being developed in the context of modal fuzzy logics, we can still obtain some results for particularly interesting conditions. Motivated by the application to preference modelling in Section 4, we will study the classes of transitive, reflexive and symmetric models, and also the class of models whose accessibility relation is *crisp* (i.e., evaluated only on $\{0,1\}$).

Even though most of the literature addresses fuzzy relations as those evaluated over the interval $[0,1]$, there is no motivation for that restriction in general, and most of the conventions, notions and results known for fuzzy relations are preserved in the more general context of relations evaluated over bounded integral residuated lattices (i.e., those where there exists a top and bottom elements that coincide with the usual constants 0 and 1). From this reflection, the definition of reflexive, $\odot$-transitive and symmetric **A**-Kripke models is immediate: an **A**-Kripke model $\mathfrak{M}$ is:

R: *Reflexive* when $R(v,v)=1$ for all $v \in W$.
S: *Symmetric* when $R(v,w)=R(w,v)$ for any $v,u \in W$.
T: $\odot$-*Transitive* when $R(v,w)\odot R(w,u) \leqslant R(v,u)$ for any $v,u,w \in W$.

If $\mathbb{P}$ denotes one or more of the previous conditions, we will denote by $(\mathbb{P})\mathbb{BM}_{\mathbf{A}}$ the class of **A**-Kripke models satisfying the conditions from $\mathbb{P}$. It is not hard to see that the well-known modal axioms that characterize the previous frame conditions in classical modal logic also characterize their corresponding many-valued counterpart defined above.

Proposition 1. *Let* X *be one or more of the following pairs of axiom schematas:*

– $(\mathsf{T}\Box)$: $\Box\varphi \to \varphi$ *and* $(\mathsf{T}\Diamond)$: $\varphi \to \Diamond\varphi$ *(reflexivity)*
– $(\mathsf{B}\Box)$: $\Diamond\Box\varphi \to \varphi$ *and* $(\mathsf{B}\Diamond)$: $\varphi \to \Box\Diamond\varphi$ *(symmetry)*
– $(\mathsf{4}\Box)$: $\Box\varphi \to \Box\Box\varphi$ *and* $(\mathsf{4}\Diamond)$: $\Diamond\Diamond\varphi \to \Diamond\varphi$ *(transitivity)*

Then let $(\mathsf{X})\mathsf{BM}_{\mathbf{A}}$ *be the axiomatic extension of* $\mathsf{BM}_{\mathbf{A}}$ *with the axioms from* X*, and let* $\mathbb{P}$ *be the model conditions corresponding to the axioms in* X*. Then, for any* $\Gamma \cup \{\varphi\} \subseteq MFm$*,* $\Gamma \vdash_{(\mathsf{X})\mathsf{BM}_{\mathbf{A}}} \varphi$ *iff* $\Gamma \Vdash_{(\mathbb{P})\mathbb{BM}_{\mathbf{A}}} \varphi$*.*

Proof. Soundness is easy to check in all three cases. Concerning completeness, it is just necessary to take into account that the canonical model for $(\mathsf{X})\mathsf{BM}_{\mathbf{A}}$ is defined in the same way as the one for $\mathsf{BM}_{\mathbf{A}}$ but taking into account the new equations arising from the additional axioms in the definition of the worlds of the model (that now need to validate them). Under this consideration, reflexivity follows immediately from the definition of $R^c(v,v)$. Indeed,

$$R^c(v,v) = \bigwedge_{\psi \in Fm} (v(\Box\psi) \to v(\psi)) \wedge (v(\psi) \to v(\Diamond\varphi),$$

and due to the reflexivity axioms $(T\Box)$ and $(T\Diamond)$ it follows that $R^c(v,v)=1$.

As for symmetry, assume towards a contradiction that for some $v,w \in W^c$, $R^c(v,w) \not\leqslant R^c(w,v)$. By definition, this means that there is some formula φ such that $R^c(v,w) \not\leqslant (w(\Box\varphi) \to v(\varphi)) \wedge (v(\varphi) \to w(\Diamond\varphi))$, and thus, at least one of the following situations must hold:

(1) $R^c(v,w) \not\leqslant w(\Box\varphi) \to v(\varphi)$,
(2) $R^c(v,w) \not\leqslant v(\varphi) \to w(\Diamond\varphi)$.

It is easy to show that any of the previous conditions leads to a contradiction with the symmetry axioms. For if (1) were to hold, necessarily we would also have $R^c(v,w) \not\leqslant w(\Box\varphi) \to v(\Diamond\Box\varphi)$ due to axiom $(\mathsf{B}\Diamond)$, but this contradicts the definition of $R^c(v,w)$ (since $R^c(v,w) \leqslant w(\psi) \to v(\Diamond\varphi)$ for all ψ). The second possible situation (2) is handled in the same way by resorting to axiom $(\mathsf{B}\Box)$.

The case of $\odot$-transitivity is a bit more cumbersome but equally simple. Assume towards a contradiction that there are some $v, w, u \in W^c$ for which $R^c(v,w) \odot R^c(w,u) \not\leqslant R^c(v,u)$. Then, by definition of $R^c(w,u)$, there is some formula φ such that $R^c(v,w) \odot R^c(w,u) \not\leqslant (v(\Box\varphi) \to u(\varphi)) \wedge (u(\varphi) \to v(\Diamond\varphi)$. As above, there are two possible cases:

(1') $R^c(v,w) \odot R^c(w,u) \not\leqslant (v(\Box\varphi) \to u(\varphi))$ or
(2') $R^c(v,w) \odot R^c(w,u) \not\leqslant (u(\varphi) \to v(\Diamond\varphi)$.

We can again show that none of the previous conditions can hold. We will show the first one, the other is done analogously (using the dual $(4\Diamond)$ axiom). Observe that by Axiom $(4\Box)$, together with the fact that $\to$ is decreasing in the first component, (1') implies that $R^c(v,w) \odot R^c(w,u) \not\leqslant (v(\Box\Box\varphi) \to u(\varphi)) = \bigwedge_{z \in W^c}(R^c(v,z) \to z(\Box\varphi)) \to u(\varphi)$. In particular (resorting again to the antimonotonicity of $\to$), letting $z = w$, we get $R^c(v,w) \odot R^c(w,u) \not\leqslant (R^c(v,w) \to w(\Box\varphi)) \to u(\varphi)$ By the residuation law it follows $R^c(v,w) \odot (R^c(v,w) \to w(\Box\varphi)) \not\leqslant R^c(w,u) \to u(\varphi)$ and thus, we get the contradictory statement $w(\Box\varphi) \not\leqslant R^c(w,u) \to u(\varphi)$. □

Interestingly enough, the previous completeness results allow us to characterize the class of models with fuzzy $\odot$-preorders (i.e. reflexive and $\odot$-transitive models) with axioms $(\mathsf{T}\Box, \mathsf{T}\Diamond)$ and $(\mathsf{B}\Box, \mathsf{B}\Diamond)$. However, if we further add axioms $(4\Box, 4\Diamond)$ we do not get an axiomatization of the class of models with a universal relation (i.e., for which $R(v,w) = 1$ for all v, w), in contrast with what happens in the classical case.

This lack of expressibility can be solved, when the underlying truth-value algebra $\mathbf{A}$ enjoys certain nice properties, by combining the previous axiomatic extensions with a system complete with respect to the models whose accessibility relation is crisp, i.e. evaluated only over $\{0,1\}$, Indeed, it is possible to provide such an axiomatic system whenever $\mathbf{A}$ is subdirectly irreducible (SI)[7], with the same approach that the one followed in [4]. For the sake of simplicity, we will focus in this paper in the particular case of $\mathbf{A}$ being a linearly ordered residuated lattice, which is always a SI residuated lattice. The key idea is the fact that any (finite) SI residuated lattice $\mathbf{A}$, and so, any linearly ordered one, has a unique coatom k, i.e. a unique element $k < 1$ such that, for any $a \in A$, if $a < 1$ then $a \leqslant k$. Since it coincides almost exactly with the proof of [4, Th. 4.22], we do not detail here the proof of the following result.

Theorem 2. *Let* $\mathbf{A}$ *be a finite linearly ordered residuated lattice, and* $\mathbb{C}_{\mathbf{A}}$ *be the class of crisp* $\mathbf{A}$*-Kripke models. Define the logic* $\mathsf{CBM}_{\mathbf{A}}$ *as the extension of* $\mathsf{BM}_{\mathbf{A}}$ *with the axiom schemata*

[7] For the interested reader, see eg. [5] for an insight on the importance of this kind of algebras.

– $(\Box\mathsf{k})$ $\Box(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box\varphi)$,

and let $\vdash_{\mathsf{CBM}_\mathbf{A}}$ *denote the corresponding notion of proof. Then, for any* $\Gamma \cup \{\varphi\} \subseteq MFm$, $\Gamma \vdash_{\mathsf{CBM}_\mathbf{A}} \varphi$ *iff* $\Gamma \Vdash_{\mathbb{C}_\mathbf{A}} \varphi$.

As a direct corollary we get the following result.

Corollary 1. *Let* $\mathsf{S5BM}_\mathbf{A}$ *be the axiomatic extension of* $\mathsf{BM}_\mathbf{A}$ *with the axioms* $(\mathsf{T}\Box), (\mathsf{T}\Diamond), (\mathsf{B}\Box), (\mathsf{B}\Diamond), (4\Box), (4\Diamond)$ *and* $(\Box\mathsf{k})$*, and consider the class* $\mathbb{U}_\mathbf{A}$ *of universal* **A***-Kripke models. Then, if* **A** *is a finite linearly ordered residuated lattice, for any* $\Gamma \cup \{\varphi\} \subseteq MFm$*, we have* $\Gamma \vdash_{\mathsf{S5BM}_\mathbf{A}} \varphi$ *iff* $\Gamma \Vdash_{\mathbb{U}_\mathbf{A}} \varphi$.

4 Modelling fuzzy preferences

In this section, as a matter of illustrating application, we show how the logical machinery developed in the previous sections can be used to devise a logical framework to represent and reason with fuzzy preferences.

We take as starting point van Benthem et al.'s work [1] where, among other logics, the authors consider a basic (classical) modal logic of weak and strict preference interpreted in ordered models of possible worlds, provide a complete axiomatization, and show how global preferences between propositions can be defined by lifting the world ordering to an ordering between sets of worlds. Actually they consider different possibilities to define such global preferences based on (crisp) preference models, i.e. structures $M = (W, \preceq, e)$, where $\preceq$ is preorder on the set of worlds and e is a valuation. The language contains two modal operators, a global S5 modality A and a S4 modality $\Box$, where $\mathsf{A}\varphi$ reads that φ is true in all the worlds, while $\Box\varphi$ reads that φ is true in all the worlds that are more preferred (in the sense of $\preceq$) than the current world. Then, one possibility to encode that "ψ is weakly preferred to φ" is by the formula

$$\varphi \leqslant_{\forall\exists} \psi := \mathsf{A}(\varphi \to \Diamond\psi),$$

to be interpreted as expressing that for any world where φ is true, there is a more preferred world where ψ holds.

In what follows we show how the above framework can be faithfully generalised to deal with both fuzzy propositions and preferences, taking values in a linearly ordered finite residuated lattice **A**. The reason to restrict ourselves to linearly ordered algebras **A** is due to the need of using a global modal operator, for which we only have an axiomatization in such a case, see previous section.

Thus, for modelling preferences we consider a language **PFm** expanding **MFm** with two additional unary operators A and E, which will the role of global operators. The intended semantics is given by the class of reflexive and $\odot$-transitive **A**-Kripke models, that we will call **A**-preference models.

Definition 4. *An* **A***-preference model* $\mathfrak{P}$ *is a triple* $\mathfrak{P} = \langle W, R, e\rangle$ *such that*

– W *is a set of worlds,*

- $R\colon W \times W \to A$ *is an* A*-valued fuzzy pre-order, i.e. a reflexive and* $\odot$*-transitive relation between worlds,*
- $e\colon W \times \mathcal{V} \to A$ *is a* **A***-evaluation of variables that is uniquely extended to formulas of* **PFm** *as in Def. 1 for the propositional connectives and operators* $\Box$ *and* $\Diamond$*, and for the new operators is extended as follows:*

$$e(v, \mathsf{A}\varphi) = \bigwedge_{w\in W} \{e(w,\varphi)\} \qquad e(v, \mathsf{E}\varphi) = \bigvee_{w\in W} \{e(w,\varphi)\}.$$

We will denote by $\mathbb{P}_{\mathbf{A}}$ the class of **A**-preference models, and use $\Vdash_{\mathbb{P}_{\mathbf{A}}}$ with the analogous meaning it had for **A**-valued Kripke models in the previous sections.

After the work developed in the previous sections, it is very natural the way to provide an axiomatic system complete with respect to $\Vdash_{\mathbb{P}_{\mathbf{A}}}$.

Theorem 3 (Completeness). *Let* $\mathsf{P}_{\mathbf{A}}$ *be the deductive system given by:*

- *The axioms and rules of* $(\mathsf{T4})\mathsf{BM}_{\mathbf{A}}$ *for the* $\Box$ *and* $\Diamond$ *operators*
- *The axioms and rules of* $\mathsf{S5BM}_{\mathbf{A}}$ *for the* A*,* E *operators*
- *The* inclusion *axiom schematas:* $\mathsf{A}\varphi \to \Box\varphi$*,* $\Diamond\varphi \to \mathsf{E}\varphi$

Denoting by $\vdash_{\mathsf{P}_{\mathbf{A}}}$ *its corresponding notion of proof, then for any* $\Gamma \cup \{\varphi\} \subseteq PFm$*, we have* $\Gamma \vdash_{\mathsf{P}_{\mathbf{A}}} \varphi$ *iff* $\Gamma \Vdash_{\mathbb{P}_{\mathbf{A}}} \varphi$*.*

Proof. Soundness is a simple exercise. As for completeness, analogously to the approach to prove completeness for the minimal bimodal logic $\mathsf{BM}_{\mathbf{A}}$ in Section 2, one can build a corresponding canonical model $\mathfrak{P}^c = (W^c, R_1^c, R_2^c, e^c)$, with $W^c := \{h \in Hom(\mathbf{PFm}, \mathbf{A})\colon h(Th(\mathsf{P}_{\mathbf{A}})) = 1\}$ and this time with two accessibility relations R_1^c and R_2^c, one for the pair of operators $(\Box, \Diamond)$ and another for the operators (A, E). By Proposition 1, it follows that R_1^c is a fuzzy $\odot$-preorder (it is reflexive and $\odot$-transitive), while R_2^c is a crisp equivalence relation. The corresponding truth-lemma (analogous to Lemma 2) shows that if $\Gamma \not\vdash_{\mathsf{P}_{\mathbf{A}}} \varphi$ there is world $v_0 \in W^c$ for which $e(v_0, \gamma) = 1$ for all $\gamma \in \Gamma$ and $e(v_0, \varphi) < 1$. Note that this does not prove yet the claim of the theorem, since R_2^c is not guaranteed to be the universal binary relation on W^c. So a bit more elaboration is needed.

Due to the inclusion axioms, it is immediate to see that $R_1^c(v, w) \leqslant R_2^c(v, w)$ for any pair of worlds, so in particular

$$\text{if } R_1^c(v, w) > 0 \text{ then } R_2^c(v, w) = 1. \tag{4}$$

At this point, we can consider the submodel $\mathfrak{P}^c_{v_0}$ generated by v_0 with respect to R_2, i.e., the model whose universe is $W^c(v_0) = \{w \in W^c\colon R_2(v_0, w) = 1\}$, whose relations are the restrictions of R_1^c and R_2^c to $W^c(v_0)$, and whose evaluation of variables is the same. We only need to check that the truth-evaluations in the submodel (in the worlds from $W^c(v_0)$) and in the original model are the same. Note that R_2^c on $W^c(v_0)$ is total. This can be proved by induction on the complexity of the formula, being immediate for the cases concerning non-modal connectives. As for the modal operators, first observe that for any $u, w \in W$, if $u \in W^c(v_0)$ and $w \notin W^c(v_0)$ it follows that $R_2^c(u, w) = 0$ (since R_2^c is an equivalence relation), and hence $R_1^c(u,w) = 0$ as well. Then, for any $u \in W^c(v_0)$:

$$e_v^c(u, \Box\varphi) = \bigwedge_{w \in W^c(v_0)} \{R_1^c(u,w) \to e^c(w,\varphi)\} = \bigwedge_{w \in W^c} \{R_1^c(u,w) \to e^c(w,\varphi)\},$$
$$e_v^c(u, \mathsf{A}\varphi) = \bigwedge_{w \in W^c(v_0)} e^c(w,\varphi) = \bigwedge_{w \in W^c : R_2^c(u,w)=1} e^c(w,\varphi).$$

A similar argument can be done for $\Diamond$ and E. This concludes the proof, since the resulting model $\mathfrak{P}_{v_0}^c$ is an **A**-preference model in which there is a world v_0 satisfying Γ and not φ. $\Box$

From the above, in the frame of the $\mathsf{P}_{\mathbf{A}}$ logic one can represent the (weak) preference of a proposition ψ over another φ by the expression $\mathsf{A}(\varphi \to \Diamond\psi)$. This preference between propositions actually enjoys the properties of a fuzzy $\odot$-preorder, which justifies in a sense the meaningfulness of this choice. Indeed,

- Reflexivity: $\mathsf{A}(\varphi \to \Diamond\varphi)$ is valid in $\mathbb{P}_{\mathbf{A}}$, since $\varphi \to \Diamond\varphi$, i.e. axiom ($4\Diamond$), is valid in $\mathbb{P}_{\mathbf{A}}$.
- $\odot$-Transitivity: one can show that

$$\mathsf{A}(\varphi \to \Diamond\psi) \odot \mathsf{A}(\psi \to \Diamond\chi) \to \mathsf{A}(\varphi \to \Diamond\chi) \tag{5}$$

  is also a valid formula in $\mathbb{P}_{\mathbf{A}}$. Namely, this follows by first showing that the following formula expressing a form of monotonicity for $\Diamond$ holds true in $\mathbb{P}_{\mathbf{A}}$:

$$\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\Diamond\varphi \to \Diamond\psi).$$

  This leads to $\mathsf{A}(\psi \to \Diamond\chi) \to \mathsf{A}(\Diamond\psi \to \Diamond\Diamond\chi)$, but since $\Diamond\Diamond\chi \to \Diamond\chi$ holds true, we get $\mathsf{A}(\varphi \to \Diamond\psi) \odot \mathsf{A}(\psi \to \Diamond\chi) \to \mathsf{A}(\varphi \to \Diamond\psi) \odot \mathsf{A}(\Diamond\psi \to \Diamond\chi)$, and by axiom K for A, it follows the validity of

$$\mathsf{A}(\varphi \to \Diamond\psi) \odot \mathsf{A}(\Diamond\psi \to \Diamond\chi) \to \mathsf{A}(\varphi \to \Diamond\chi),$$

  that directly allows us to show the validity of (5).

5 Conclusions

In this paper we have been concerned with completing the notion of the minimal modal logic over a finite residuated lattice (and with truth-constants) from [4] to get a full-fledged modal logic with both a necessity and a possibility operators. The gain in expressibility has been used, as a matter of example, to define a many-valued counterpart of a modal logic studied in [1] to reason about preferences between propositions in such a many-valued setting. As for future work, there are several interesting open research issues that are left open, among them:

- axiomatization of modal expansions of logics arising from varieties generated by a finite residuated lattice, probably without resorting to canonical truth-constants;

– better understanding of the expressibility of general frame/model conditions in residuated lattice-based modal logics;
– deepening on the axiomatization of the logic arising from crisp Kripke models over non SI residuated lattices;
– general study of a larger set of preference relations definable in the many-valued context introduced in this work, along the line of [1, 13].

Acknowledgements. Vidal acknowledges support by the joint project Austrian Science Fund (FWF) I1897-N25 and Czech Science Foundation (GACR) 15-34650L, and by the institutional grant RVO:67985807. Esteva and Godo acknowledge support by the FEDER/MINECO project TIN2015-71799-C2-1-P.

References

1. van Benthem, J., Girard, P., Roy, O.: Everything else being equal: A modal logic for ceteris paribus preferences. Journal of Philosophical Logic **38**, 83–125 (2009)
2. van Benthem, J., Liu, F.: Dynamic Logic of Preference Upgrade. Journal of Applied Non-Classical Logic, 17(2), 157–182 (2007).
3. Bou, F., Cerami, M., Esteva, F.: Finite-valued Łukasiewicz modal logic is PSPACE-complete. In: Proc. of IJCAI 2011, pp. 774–779 (2011).
4. Bou, F., Esteva, F., Godo, L., Rodríguez, R.: On the minimum many-valued modal logic over a finite residuated lattice. Journal of Logic and Computation **21**(5), 739–790 (2011).
5. Burris, S., Sankappanavar, H.P.: A course in Universal Algebra, the Millennium, 2012 update edn. (2012).
6. Caicedo, X., Metcalfe, G., Rodriguez, R., Rogger, J.: Decidability of order-based modal logics. Journal of Computer and System Sciences (2016). In Press.
7. Caicedo, X., Rodríguez, R.O.: Standard Gödel modal logics. Studia Logica **94**(2), 189–214 (2010).
8. Caicedo, X., Rodríguez, R.O.: Bi-modal Gödel logic over $[0, 1]$-valued kripke frames. Journal of Logic and Computation **25**(1), 37–55 (2015).
9. Cintula, P., Noguera, C., Rogger, J.: From kripke to neighborhood semantics for modal fuzzy logics. In: J. Carvalho et al. (eds.) Proceedings of IPMU 2016, Part II, CCIS 611, pp. 123–135 (2016).
10. Fan, T.F.: Fuzzy bisimulation for Gödel modal logic. IEEE Trans. Fuzzy Systems **23**(6), 2387–2396 (2015)
11. Fan, T.F., Liau, C.J.: Logical characterizations of regular equivalence in weighted social networks. Artificial Intelligence **214**, 66–88 (2014)
12. Hansoul, G., Teheux, B.: Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. Studia Logica **101**(3), 505–545 (2013).
13. Liu, F.: Reasoning about Preference Dynamics. Vol. 354 of the Series Synthese Library, Springer-Verlag, 2011.
14. Marti, M., Metcalfe, G.: A hennessy-milner property for many-valued modal logics. In: R. Goré, B.P. Kooi, A. Kurucz (eds.) Advances in Modal Logic 10, pp. 407–420. College Publications (2014)
15. Priest, G.: Many-valued modal logics: a simple approach. The Review of Symbolic Logic **1**(2), 190–203 (2008).
16. Vidal, A.: On modal expansions of t-norm based logics with rational constants. Ph. D. Dissertation, University of Barcelona (2015).
17. Vidal, A., Esteva, F., Godo, L.: On modal extensions of product fuzzy logic. Journal of Logic and Computation **27**(1), 299–336 (2017).